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ON THE FAIR AND EFFICIENT ALLOCATION OF INDIVISIBLE COMMODITIES--ETC(U)

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ON THE FAIR AND EFFICIENT ALLOCATION OF INDIVISIBLE COMMODITIES

by

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BIOGRAPHICAL SKETCH

Born on August 28, 1952, Richard Engelbrecht-Wiggans attended Harvard College and received an A.B. degree in chemistry and physics in 1973. After working one year as a consultant to various computer software firms, he started graduate work in Operations Research at Cornell University, and received a Ph.D. in the winter 1977. He is currently employed as a research economist with the Cowles Foundation at Yale University.

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NOTATION

$N = \{1,2,\ldots,n\}$ is the set of individuals or players.

$M = \{1,2,\ldots,m\}$ is the estate or set of goods to be allocated.

D is the amount of "dollars" in the original estate (usually zero).

$\underline{s} = (s_1,s_2,\ldots,s_n)$ is an assignment of the m goods among the players. The s_i represent a partition of M. $\underline{s}'$ and $\underline{s}^*$ will also be used if several assignments have to be discussed at the same time.

$\underline{a} = (a_1,a_2,\ldots,a_n) = (s_1+d_1,\ s_2+d_2,\ldots,s_n+d_n)$ is an allocation of the estate $M + D$; where $\sum_{i=1}^{i=n} d_i = D$. In general, "assignment" refers to a partition of the goods in M, while "allocation" refers to an assignment together with side payments which sum to D. $\underline{a}'$ and $\underline{a}^*$ will also denote allocations.

A is the set of all possible allocations.

A^* is the set of Pareto elements of A above.

$u_i(a_i) = u_i(s_i + d_i)$ is the utility function of player i for the award $a_i = s_i + d_i$. In general, both "+" and "$\cup$" will be used to indicate the union of two subsets. The amount of dollars may be prefaced by "\$" if it is desired to emphasize the fact that this term represents "dollars." Thus, $s_i + d_i$, $s_i \cup \$d_i$, and $s_i + \$d_i$ are alternate expressions for the same quantity.

$v_i(a_i) = v_i(s_i+d_i)$ is the dollar value of the set a_i to player i. As with utilities, "+" and "$\cup$" are used interchangeably, and the dollars may be prefaced by "\$."

$R(\underline{s}) = \sum_{i=1}^{i=n} v_i(s_i)$ is the revenue realized from auctioning the goods according to the assignment $\underline{s}$.

R^* is the maximum, over all possible assignments, of $R(\underline{s})$.

$E(\underline{s})$ is the "excess" revenue associated with using the assignment $\underline{s}$ in some particular allocation scheme. The definition of the excess varies from one scheme to another.

f_i is the share player i has in the estate.

w_i is the share player i has in any excess.

$\#(s)$ denotes the cardinality of the set s.

$Int(x)$ is the largest integer not exceeding x.

$D_i(j)$ is a non-negative "discount" function.

$(N_1,N_2,\ldots,N_k)$ is a partition of the players N into coalitions. However, N_i denotes merely the collection of players in N_i; it does not imply that these players form a coalition.

$n_i = \#(N_i)$ is the number of players in N_i.

$f_{N_i} = \sum_{j:\ j \in N_i} f_j$ is the share of N_i in the estate.

$w_{N_i} = \sum_{j:\ j \in N_i} w_j$ is the share of N_i in any excess.

$v_{N_i}(s)$ is the value of a revenue maximizing assignment of the goods in s among the players in N_i.

$v_i'(s)$ is the value player i actually bids on the set s. The bid functions v_i' are assumed to be additive in dollars.

$v_{N_i}'(s)$ is the value of a revenue maximizing assignment of the goods in s among players in N_i when the functions v_i' are used in place of the true values v_i.

$s_{N_i} = \cup_{j:\ j \in N_i}\ s_j$ is the collection of goods assigned by the assignment $\underline{s}$ to players in N_i.

$a_{N_i} = \cup_{j:\ j \in N_i}\, a_j$ is the total award under the allocation $\underline{a}$ to the players in N_i.

$R'(\underline{s}) = \sum_{i=1}^{i=n} v_i'(s_i)$.

$R^{*\prime}$ = the maximum, over all possible assignments, of $R'(\underline{s})$.

$e(s) = v_{N_i}'(s) - v_{N_i}(s)$ is the amount by which N_i overbids the set s of goods. The N_i used will be clear from the context. Throughout the chapter on strategies, "e" will be used as an abbreviation for e(M+D) (which is equal to e(M), since the value functions are assumed to be additive).

d,e, and δ generally represent small positive quantities. However, in any particular context, these symbols may be defined much more specifically. Occasionally, "d" and "e" will be used in other contexts; e.g., e = e(M) and d = max(0,e).

CHAPTER I

THE FAIR DIVISION PROBLEM

I.1 Introduction

The world abounds with examples of fair division problems. Starting with our childhood, we concerned ourselves with questions of how to fairly divide a bag of candy among several children, who should do what household chore, and deciding what we would do with our time today. As adults, questions of allocating tax revenues or dividing an inherited estate have more relevance. Selling off shore oil leases and dividing each year's athletic talent among the professional football teams are yet more examples of fair division problems.

"Divide and choose" schemes are particularly well suited to allocating a homogeneous finely divisible commodity. Two children desiring to split equally a bag of small candies might have one divide the candies into two approximately equal piles and let the second choose one of the piles. Although such schemes may be extended to allocating inhomogeneous or indivisible commodities and to more than two people, the results are not always satisfactory in terms of fairness or Pareto optimality.

To illustrate an unsatisfactory aspect of applying divide and choose schemes to inhomogeneous items, consider the case of two college students trying to split a pizza "in half". Assume that half the pizza is covered with sausage, whereas the other half is left plain. The symmetric, and perhaps instinctive, solution is two pieces each consisting of equal amounts of plain and sausage pizza.

However several difficulties arise because of the common and reasonable situation where different individuals have different preferences.

If one student has a strong preference for sausage pizza, while the other has a mild preference for plain, then both students would be happier if one received all of the sausage pizza and the other all the plain. Thus, the symmetric solution is not Pareto optimal or "efficient".

There is an asymmetry among individuals in the divide and choose scheme; one divides while the other chooses. This asymmetry along with the differing preferences enables the divider to take advantage of the chooser in situations where the divider knows the chooser's preferences. The student preferring plain pizza could divide the pizza into pieces such that one consists of all the plain and some of the sausage pizza and still be confident that the other student will choose the piece consisting of the remaining sausage pizza.

If the items are not finely divisible, then the required division into piles may be impossible. Luce and Raiffa [15; page 367] propose that each individual then adds a sufficiently large amount of money to the set of items, and then proceed as before. However, once a transferable commodity, namely dollars, is introduced there are many alternatives to the divide and choose schemes.

This paper considers the case in which there exists a finely divisible tranferable commodity. Although this commodity may be any of a variety of things, such as, sand, hours of committee work, or conventional money, it will be referred to as dollars. Using the individuals' valuations of the items, it will be possible to devise efficient allocation schemes; schemes which are also symmetric with respect to the players.

Upon defining the problem under consideration more precisely and after establishing some notation, the concept of value will be discussed.

Since the transferable commodity plays a central role in the fair division problem, any assumptions (and implications of these assumptions) will be examined closely. It is shown how the value functions preserve the preference orders imposed on the items by the players' utility functions. Using the preference ordering recorded by the value functions, Pareto optimality is defined and it is shown that the Pareto optimal allocations may be achieved by auctioning the estate in a way to maximize revenue and then divide the revenue among the players.

Implicit in traditional auctions and fair division schemes is the assumption that players' values be additive in all goods. Some of the implications and shortcomings of this assumption are examined, and the fair allocation scheme of Knaster is presented. Although this scheme implicitly assumes that all players' value functions are additive in all commodities, it will later be used to help motivate some of the fair division schemes under less restricted value functions.

Since not all Pareto optimal allocations should be considered fair to all players, it is necessary to establish what collection of subsets of the estate each player should reasonably consider fair. These fair sets tend to be characterized by a number for each player. Any subset of the estate worth at least this number should be considered fair by the appropriate player.

Kuhn's definition for fairness is presented and used to determine a collection of fair sets. The resulting fair allocations include some non-pareto optimal allocations and require all players to have an equal share in the estate. Several modifications, including that of Dubins and Spanier, are examined as possible solutions to these difficulties. However, as some examples illustrate, none of these modifications results

in a totally satisfactory fairness definition.

The choice of a finely divisible transferable commodity to play the role of dollars is not without its consequences. If one considers the question of a fair share both with respect to the original estate and with respect to the, perhaps extraneous, commodity used for dollars then several ambiguities arise; it is difficult to require that individuals receive a fair share with respect to both. This paper usually considers fairness defined in terms of the distinguished transferable commodity. Different choices of what commodity is to play the role of dollars may affect what allocations are considered fair.

This chapter concludes with a fair allocation scheme of Dubins which allows a player to express preferences on how the goods the remaining players receive are assigned among these players.

I.2 Description of Problem

A collection of m items is to be allocated among n individuals. The individuals, hereafter called players, may be beneficiaries dividing an inherited estate, students splitting a pizza, professional teams drafting athletes, or firms bidding for government contracts. The players need not be human; the problem may be to allocate one's time among several tasks.

Although this paper does not require that the items be indivisible, consideration of divisible commodities requires additional restrictions to assure measurability of subsets, existence of maxima, and similar technicalities. The determined reader is encouraged to keep the possibility of divisible commodities in mind. However, in order not to obscure

the thrust of this paper with the details required for complete generality, the items will hereafter be considered to be distinct indivisible commodities.

It is assumed that there is a distinguished finely divisible transferable commodity called dollars. This commodity will be used to assess players' values for sets of items. Side payments among players may be used to compensate for the fact that the commodities are indivisible and can not be divided among players.

The objective is to identify Pareto optimal, or efficient, allocations. From these, a fair or equitable allocation must be chosen. In order to define the problem more precisely, some notation must be established.

Define the following:

$N = \{1,2,\ldots,n\}$ = set of players;

$M = \{1,2,\ldots,m\}$ = set of goods to be allocated;

D = any dollars in the estate (usually equal to zero);

$\underline{s} = (s_1,s_2,\ldots,s_n)$ = an assignment of the m items (a partition of M into n disjoint sets s_i);

$\underline{a} = (a_1,a_2,\ldots,a_n) = (s_1+d_1,s_2+d_2,\ldots,s_n+d_n)$, = an allocation of the estate $M + D$; where $\sum_{i=1}^{i=n} d_i = D$,

A = set of all possible allocations of $M + D$; and

f_i = share of player i in the estate; $\sum_{i=1}^{i=n} f_i = 1$, $f_i \geq 0$.

For each player i in N, let $u_i(s_i+d_i)$ be the von Neumann-Morgenstern [28; Third Edition, Appendix] utility function of player i for subsets s_i of M and all real values of d_i. The utility functions

are assumed to be continuous, strictly increasing, and unbounded functions of dollars. It is now possible to define the concept of value.

I.3 Values; Assumptions and Implications

Since money is the distinguished finely divisible transferable commodity, it will be useful to determine the dollar equivalent for sets of items. For each player i in N, define the value function $v_i(s_i+d_i)$ as the real number x such that $u_i(s_i+d_i)$ is equal to $u_i(\emptyset+\$x)$ where $\emptyset$ denotes the empty set. The value x may be thought of as the dollar value of the set s_i together with d_i dollars.

The requirement that u_i is continuous and unbounded in dollars assures that such an x exists. Since the utility function is strictly increasing, there will be a unique value for x. For future reference, note that $u_i(\$v_i(s_i+d_i)) = u_i(s_i+d_i)$ for all sets s_i and all values of d_i.

Recall, though dollars is the name given to the distinguished transferable commodity, the dollars may be sand, time, or any other finely divisible commodity. Thus $v_i(s_i+d_i)$ would be the "sand equivalent" of s_i+d_i for player i if sand is chosen to be the distinguished commodity. This more general concept of value applies throughout the paper even though the exposition is in terms of "dollars."

The choice of a commodity to serve as dollars may have some effect on what allocations are considered equitable. A subsequent example will illustrate this more completely. Fortunately, in many situations either conventional money or some other readily identified commodity is the "natural" choice. The question of how to decide which commodity to use will not be addressed in this paper.

It is necessary to place some restrictions on the value functions. In order to compare different players' values for sets of items, it will be assumed that $v_i(\emptyset+0)$ is equal to zero for all players. This assumption does resemble interpersonal comparison of utilities (for the null set) and is therefore not totally innocent. However, some scaling assumption is necessary, and the above is one commonly used in the real world. Alternatives will be discussed briefly later.

In addition, it will be assumed that $v_i(s_i+d_i)$ is equal to $v_i(s_i) + d_i$ for all sets s_i and all values of d_i. This requires that an individual's value for a set of items be independent of the individual's wealth, and has some strong implications. If the estate may include arbitrary lotteries for dollars, then according to Raiffa [25, page 90], the assumption that value functions are additive in dollars forces the players to have utility functions which are either linear or exponential. More precisely, the assumption implies that $u_i(\emptyset+\$d_i)$ is either of the form cd_i or else of the form $1 - \exp(-d_i/c)$.

This implication is clearly quite restrictive. It will be shown later that most of the results still hold if the additivity assumption is relaxed to the much less restrictive, and intuitively plausible, assumption that $v_i(s_i+d_i)$ is a strictly increasing function of d_i for each set s_i and each player i. But, since the more restrictive form greatly simplifies the analysis, it will be used throughout most of the paper.

It should be noted that the value functions preserve the preference order among sets resulting from the utility functions, as shown by the following two lemmas.

<u>Lemma I.1:</u> $u_i(a_i) \geq u_i(b_i)$ iff $v_i(a_i) \geq v_i(b_i)$, and $u_i(a_i) = u_i(b_i)$ iff $v_i(a_i) = v_i(b_i)$.

Proof: The proof for each statement follows directly from the definitions and assumptions. Since u_i is a strictly increasing function of dollars, $v_i(a_i) \geq v_i(b_i)$ implies that $u_i(v_i(a_i)) \geq u_i(v_i(b_i))$. Since, by definition, $u_i(v_i(x)) = u_i(x)$, this implies that $u_i(a_i) \geq u_i(b_i)$. Likewise, $v_i(a_i) < v_i(b_i)$ implies that $u_i(a_i) < u_i(b_i)$. The remainder of the proof is trivial.

<u>Lemma I.2:</u> Let L^1 and L^2 be lotteries which give the goods L^j_k with probability $p^j_k(\sum_k p^j_k = 1$ for $j = 1,2)$. If $\min_{L^1_k \in L^1} v_i(L^1_k) \geq \max_{L^2_k \in L^2} v_i(L^2_k)$ then $v_i(L^1) \geq v_i(L^2)$.

Proof: Since the least preferred outcome in L^1 has at least the value (and therefore, at least the utility) of the most preferred outcome in L^2, the expected utility of L^1 (and therefore the value of L^1) must be at least that of L^2.

It is now possible to define dominance and Pareto optimality of allocations in terms of values. Given two allocations $\underline{a} = (a_1,a_2,\ldots,a_n) \in A$ and $\underline{b} = (b_1,b_2,\ldots,b_n) \in A$, then $\underline{a}$ dominates $\underline{b}$ with respect to A (or, $\underline{a}\ \mathrm{dom}_A\ \underline{b}$) if and only if $v_i(a_i) \geq v_i(b_i)$ for all players i in N, and there is at least one player j in N such that $v_j(a_j) > v_j(b_j)$. An allocation $\underline{a}$ in A is Pareto optimal in A if there does not exist any $\underline{b}$ in A such that $\underline{b}\ \mathrm{dom}_A\ \underline{a}$.

<u>I.4 Pareto Optimal Allocations</u>

Having defined Pareto optimality for allocations, it remains to

characterize the set A^* of Pareto optimal allocations. It will be shown that Pareto optimal allocations correspond to the allocations obtained by auctioning the estate in a manner to maximize revenue and then dividing the resulting revenue among players. With such a result, the efficient and equitable allocation problem may be viewed in two parts; first the problem of efficient, or Pareto optimal, auctions, and then the problem of equitable, or fair, division of the resulting revenue.

Since the auction plays a central role in the problem, it should be defined precisely. The auction scheme which is used is a generalization of the traditional sealed bid auction suggested independently by many individuals, including Heath [10] and Vickrey [27]. Whereas, in the traditional auction each bidder submits a sealed bid on each item, in the extended scheme each bidder submits a sealed bid $v_i(s_i)$ on each subset s_i of the goods. The auctioneer is considered a bidder and also submits bids; these bids may be interpreted as reservation prices, or prices below which the particular subset of goods will not be sold.

Once the sealed bids have been submitted, they are opened and the goods are sold (or "assigned") according to the assignment $\underline{s}^*$ which maximizes the total revenue $R(\underline{s}) = \sum_{i=1}^{i=n} v_i(s_i)$ over all possible assignments. Each player i buys ("is awarded") the goods s_i at the price $\$v_i(s_i)$; this corresponds to selling each item to a high bidder at the high bid in the traditional sealed bid auction. Thus, at the end of the auction, a total revenue of $R(\underline{s})$ has been generated and the goods in M have been assigned according to the assignment $\underline{s}$. In general, it will be assumed that only revenue maximizing assignments $\underline{s}$ will be considered; such assignments will be shown to result in Pareto optimal allocations.

Neither the traditional sealed bid auction, nor the more general form, requires that the estate consist solely of indivisible commodities. If the estate contains divisible items, then it is possible for players to specify values $v_i(s_i)$ which are functions of continuously varying subsets of the divisible goods. In particular, assume the estate consists of m_1 indivisible items and $m_2 = m - m_1$ divisible commodities. If in addition, as will be assumed, the divisible commodities are homogeneous, it appears reasonable to summarize the subset s_i of goods by a vector $x^i = (x_1^i, x_2^i, \ldots, x_m^i)$ where x_j^i is the fraction of good j in the subset s_i; for indivisible goods j, x_j^i is either zero or one. (By considering only the "fraction" of any divisible good, non-measurable and otherwise pathological subsets of the estate are removed from consideration). Thus, the value function v_i is a function from $\{0,1\}^{m_1} x [0,1]^{m_2}$ to the real numbers. A revenue maximizing assignment is now a set $x^1, x^2, \ldots, x^n$ of points in $\{0,1\}^{m_1} x [0,1]^{m_2}$ with $\sum_{i=1}^{i=n} x_j^i = 1$ for all j which maximizes the function $R(\underline{x}) = \sum_{i=1}^{i=n} v_i(x^i)$ (it is assumed that the functions v_i are sufficiently regular (e.g. continuous in all divisible commodities) such that there is actually some assignment which has the value of the supremum over all assignments of $R(\underline{x})$). Throughout the remainder of the paper, the discussion will be in terms of assignments $\underline{s}$ and revenue $R(\underline{s})$ even though the concepts apply equally well to problems with some homogeneous divisible commodities.

In the above description of the generalized auction scheme, the terms "bid" and "value" have been used interchangeably. In general, however, there is no reason to expect that bidders will necessarily bid their true values. In the second and third chapters it is assumed that the v_i represent the individuals true values; either the individuals bid honestly

or the true values are determined by some other means. In these chapters, the terms "bid" and "value" will be used interchangeably since the implicit assumption is that only true values are being considered. In the last chapter, strategic aspects will be considered and there will be a difference between "bids" and "true values."

Finally, before proceeding with the discussion of efficient allocations, a simple example will be presented and used to illustrate the general auction scheme used through this paper. The example used will appear again later, and thus the actual data deserve some comment; $v_i(\emptyset) = 0$ as required by assumption, the two players have very similar value functions, and both players' value functions are slightly subadditive. It might be reasonable to expect such data in actual problems.

Example I.1: Three indivisible items (A,B, and C) will be auctioned among two players (the auctioneer and one bidder) with their respective value functions as given below.

$s =$	$\emptyset$	A	B	C	$A \cup B$	$A \cup C$	$B \cup C$	$A \cup B \cup C$
$v_1(s) =$	0	10	11	12	17	18	22	28
$v_2(s) =$	0	9	12	13	18	19	20	28

If A is sold to the first player and $B \cup C$ to the second player, the resulting revenue is $v_1(A) + v_2(B \cup C) = 10 + 20 = 30$. If player one is sold $B \cup C$ and the second player is sold A, then the resulting revenue is 31. It is easy to verify that this maximizes the total revenue. Thus $\underline{s}^* = (B \cup C, A)$ and $R(\underline{s}^*) = 31$. In other examples, it is possible that two different assignments result in the same maximizing revenue, and thus

for notational convenience, R^* will denote the value of $R(\underline{s})$ for any revenue maximizing assignment $\underline{s}$. (If some of the commodities are divisible, then there must be some regularity of the value functions in order to assure the existence of a maximum revenue.)

Using this auction it is possible to characterize the set of Pareto optimal allocations.

<u>Theorem I.1:</u> The set A^* of Pareto optimal allocations is the set of allocations resulting from auctioning the goods according to a revenue maximizing assignment and then dividing the resulting revenue, plus the D dollars (if any) part of the estate, among the players. More precisely, A^* is the allocations $\underline{a} = (a_1, a_2, \ldots, a_n) =$ $(s_1 - v_1(s_1) + d_1,\ s_2 - v_2(s_2) + d_2, \ldots, s_n - v_n(s_n) + d_n)$ for some revenue maximizing assignment $\underline{a}$ (in other words, an *assignment* $\underline{a}$ such that $\sum_{i=1}^{i=n} v_i(s_i) = R^*$) and some set of arbitrary real numbers d_i such that $\sum_{i=1}^{i=n} d_i = R^* + D$.

Proof: Such allocations $\underline{a}$ are clearly elements of A. It must be shown that these allocations are Pareto optimal, and that they include all Pareto optimal allocations.

By contradiction, assume there exists an allocation $\underline{a}'$ in A such that $\underline{a}'$ dominates $\underline{a}$ with respect to A. Then $v_i(a'_i) \geq v_i(a_i)$ for all players i and $v_j(a'_j) > v_j(a_j)$ for at least one of the players j. Thus $\sum_{i=1}^{i=n} v_i(a'_i) > \sum_{i=1}^{i=n} v_i(a_i)$. Now write $a'_i = s'_i + r_i$ where r_i is the amount of dollars in a'_i. Since a'_i is an allocation, by definition, $\sum_{i=1}^{i=n} r_i = D$. Now,

$$
\begin{aligned}
\sum_{i=1}^{i=n} v_i(s'_i) &= \sum_{i=1}^{i=n} v_i(s'_i+r_i) - D \\
&= \sum_{i=1}^{i=n} v_i(a'_i) - D \\
&> \sum_{i=1}^{i=n} v_i(a_i) - D \\
&= \sum_{i=1}^{i=n} v_i(s_i-v_i(s_i)+d_i) - D \\
&= \sum_{i=1}^{i=n} (v_i(s_i) - (v_i(s_i)-d_i)) - D \\
&= R^* - (R^* - (R^*+D)) - D \\
&= R^*.
\end{aligned}
$$

Taking the first and last expressions, $\sum_{i=1}^{i=n} v_i(s'_i) > R^*$, which is impossible. Thus $\underline{a}$ must be Pareto optimal.

For the second part of the proof, again by contradiction, let $\underline{a}' = (s'_1+r_1, s'_2+r_2,\ldots,s'_n+r_n)$ (using the same notation as above) be a Pareto optimal allocation such that $\sum_{i=1}^{i=n} v_i(s'_i) = R^* - e$ for some $e > 0$. Then, for any revenue maximizing assignment $\underline{s}$, construct the allocation $\underline{a}$ by using $a_i = s_i - v_i(s_i) + r_i + e/n$. It is easy to verify that $\underline{a}$ is indeed an allocation by using the fact that $\sum_{i=1}^{i=n} r_i = (R^* - e) + D$, and $\sum_{i=1}^{i=n} v_i(s_i) = R^*$ to verify $\sum_{i=1}^{i=n} (-v_i(s_i) + r_i + e/n) = D$ as required for an allocation. Observing that $\underline{a}$ dominates $\underline{a}'$ gives the desired contradiction.

It is clear from the above theorem that A^* itself is not a very useful solution concept for equitable allocations. Indeed, the only restriction on the d_i is that their sum be D. Some of the d_i may be negative, and therefore so may some of the $v_i(a_i)$, which are equal to $v_i(s_i-v_i(s_i)+d_i) = d_i$. This is not satisfactory for arbitrary estates. In what follows, further restrictions will be placed on the d_i.

I.5 Additive Value Functions

The usual sealed bid auction, with reservation prices, in which the high bid wins (at the high bid price) is a special case of generalized auction used in this paper. If all players' value functions are additive in all goods (i.e., $v_i(a_i \cup b_i) = v_i(a_i) + v_i(b_i)$ for all disjoint subsets of the estate) and all goods are indivisible then the profit maximizing assignments are obtained by selling each item to the highest bidder for that item.

This situation is quite different from conditions under the generalized auction. Requiring additivity of value functions in all goods is much more restrictive than only requiring the value functions to be additive on the dollar coordinate. The resulting assignment scheme is not valid for the general case; note that in example I.1 the revenue maximizing assignment did not give any item to the high bidder for the item.

This additivity in all goods assumption, implicit in traditional sealed bid auctions, is a very strong assumption. In particular, since it requires the value functions to be additive in dollars, the underlying utility functions must be either linear or exponential functions of dollars. The following lemma indicates that additive values and linear utility for dollars implies that the utility function is also additive in all commodities.

Lemma I.3: $v_i(a_i \cup b_i) = v_i(a_i) + v_i(b_i)$ for two disjoint subsets a_i and b_i of the estate (which may contain divisible commodities) and $u_i(\emptyset + d_i) = d_i\ u_i(\emptyset + \$1)$ for all real values of d_i imply that $u_i(a_i \cup b_i) = u_i(a_i) + u_i(b_i)$.

Proof: The above assumptions together with the fact that by the definition of values $u_i(\$v_i(x)) = u_i(x)$ imply

$$
\begin{aligned}
u_i(a_i \cup b_i) &= u_i(\$v_i(a_i \cup b_i)) \\
&= u_i(\$v_i(a_i) + \$v_i(b_i)) \\
&= (v_i(a_i) + v_i(b_i))\, u_i(\emptyset+\$1) \\
&= v_i(a_i)\, u_i(\emptyset+\$1) + v_i(b_i)\, u_i(\emptyset+\$1) \\
&= u_i(\$v_i(a_i)) + u_i(\$v_i(b_i)) \\
&= u_i(a_i) + u_i(b_i) \quad \text{as desired.}
\end{aligned}
$$

A considerable drawback of additive utility functions is the implication on the individual's attitude towards risk. Indeed, for otherwise arbitrary utility functions, additivity is equivalent to requiring the individual to be multivariate risk neutral [8]. For example, a multivariate risk neutral individual would be indifferent between the following two lotteries:

L^1: (painting + \$1000) and ($\emptyset$) each with probability 1/2,

L^2: (painting) and (\$1000) each with probability 1/2.

It should not be difficult to find individuals who are not indifferent between the above lotteries. If this is not convincing, assume that the painting is the Mona Lisa and that the \$1000 is increased to two million dollars. In general it would be desired not to require utilities to be additive.

The lemma shows that additive values and linear utility results in

some undesirable restrictions on the utility functions and on an individual's attitude towards multivariate risk. However, the alternate possibility of exponential utility functions does not necessarily have such unpleasant implications.

Lemma I.4: If the estate contains statistically independent lotteries over dollars and a player's utility function for dollars is exponential then the player's value for any collection of such lotteries is the sum of the individual values of the lotteries.

Proof: If $u_i(x) = 1 - \exp(-x/c)$ and L^1 and L^2 are statistically independent lotteries with outcomes represented by the random variables X_1 and X_2, then

$$
\begin{aligned}
v_i(L^1 \cup L^2) &= -c \ln E(\exp -(X_1+X_2)/c) \\
&= -c \ln E((\exp -X_1/c) \exp -X_2/c)) \\
&= -c \ln (E(\exp - X_1/c)\ E(\exp -X_2/c)) \\
&= -c \ln E(\exp - X_1/c) - c \ln E(\exp -X_2/c) \\
&= v_i(L^1) + v_i(L^2) \quad \text{as desired.}
\end{aligned}
$$

By induction, the above proof extends to sets of more than two lotteries.

Although additive values may be reasonable in some cases, there are many situations in which additive values fail to represent true values accurately. If the value functions are approximated by additive functions, the problem may be changed significantly. For example, when individuals are bidding for defense contracts, each individual would prefer to receive a few contracts. Too few contracts may mean bankruptcy, while

too many contracts may exceed the contractor's facility capacity.

In such instances, the value function may be very nearly additive for small numbers of contracts. The value for the entire collection is probably significantly less than the sum of the individual values...the additivity assumption may be a very poor approximation. The usual sealed bid auction might award a large number of bids to a small bidder who values single items highly. To remedy this situation, the bidder must resell some awards, or the small bidder must hedge (by under-bidding) on the original bids. Small companies often submit smaller bids at offshore oil leasings or must resell some of the awards [12]. It is not so much that "Sales to oil firms are 'rigged'" [4] as the fact that traditional auctions are inefficient for allocating a large number of expensive or very risky items.

Additive values can not reflect any economies, or diseconomies, of scale which may be present. A company specializing in developing computer software might discover several clients whose requirements may be met by appropriately modified versions of a single powerful software package. Satisfying each client's requirements costs only the appropriate modifications and a share of the basic development. This is likely to be less than the cost of satisfying each client's requirements independently. Thus both the software developers and the clients might profit from such a consolidation.

On the other hand, the developer might wish to avoid being awarded too many simultaneous short term contracts. When the clients are independent and distinct firms, it may be difficult to bid on sets of contracts rather than individual contracts. In some situations, such as with defense

contracts or government procurements, a large number of related contracts may be let by the same source. For such situations, a more flexible scheme is desirable.

I.6 Fair Allocations of Knaster

When the estate consists solely of indivisible items and all players have value functions additive in all commodities, then Knaster's [26] fair allocation scheme may be used to select an allocation from among the Pareto optimal allocations. It was noted before that when all value functions are additive in all goods, then a revenue maximizing assignment is obtained by selling each item to the high bidder on that item.

Knaster suggests selling each item to the high bidder, and returning to each player i the appropriate share of that player's value for the item. Any revenue remaining is divided among the players in proportion to their shares. Specifically, let f_i be the share player i has in the estate, where it is assumed that the f_i are non-negative numbers which sum to one. Now consider the following example:

Example I.2: Let two individuals have equal shares of one half in an estate consisting of three indivisible goods. Let the v_i be additive in all goods, and be as specified below for single items.

$s =$	$\emptyset$	A	B	C
$v_1(s) =$	0	10	11	12
$v_2(s) =$	0	9	12	13

According to Knaster, item A should be sold to the first player for \$10. Of that \$10, one half of $v_1(A)$ is returned to the first player and one half of $v_2(A)$ is given to the second player. Thus \$5 is returned to player one, and \$4.50 is given to player two. The remaining excess of \$10 - (\$5 + \$4.5) = \$.50 is split proportionally among the players. The first player has now received A at the cost of (\$10 - \$5.25) = \$4.75, and the second player has been given \$4.75. Notice that each player's award exceeds one half of that player's value for A.

The above procedure is repeated for the remaining items, with the final result that A is sold to the first player, B and C are sold to the second player, and the first player receives \$7.25 from the second player. Since the value functions are additive, the same result is obtained by selling all the items, returning $f_i\ v_i(A \cup B \cup C)$ to each player i and then dividing the excess in proportion to the players shares. Notice that since each item x is sold to the high bidder, say player i^*, only $\sum_{i=1}^{i=n} f_i\ v_i(x)$ of the $v_{i^*}(x)$ is returned to the players immediately. But $v_i(x) \leq v_{i^*}(x)$ for all players i and the fact that the f_i are non-negative numbers summing to one implies that $\sum_{i=1}^{i=n} f_i\ v_i(x) \leq v_{i^*}(x)$. Thus the "excess" is always nonnegative, and each player will receive more than that player's share of the player's perceived value of the estate. This scheme will later be used to motivate several different choices of a "fair" allocation from among the Pareto optimal allocations associated with generalized auctions.

I.7 A Characterization of Fair Sets

Since it is not reasonable to consider all Pareto optimal allocations to be equitable, some concept of fairness is needed. Ideally there would

exist allocations which are considered fair by all players. However, for this to be possible, there must be some restrictions on what players demand. Clearly, if each player is allowed to consider nothing less than the entire estate a fair share, then there can not exist any allocation considered fair by all players. Thus it is necessary to define under what conditions players should consider an allocation fair.

If F_i is a collection of subsets of the estate which player i considers a fair share, then it is reasonable to assume that if a_i is considered fair and there is some b_i such that $v_i(b_i)$ is at least $v_i(a_i)$, this b_i will also be considered fair. It will be assumed that the set of fair subsets is monotonic in this sense. If the collection F_i has a maximal valued element, with value C_i, then F_i is precisely the collection of subsets with value at least C_i.

It is possible that F_i does not have a maximal valued element. In this case, F_i is the collection of subsets with value strictly less than the supremum C_i of the values of elements of F_i. In either case there is some C_i which helps characterize the fair set. For many for the fairness definitions it will be convenient to use the C_i in defining fairness.

I.8 The Fairness Definition of Kuhn

Kuhn [14] suggests the following definition of fairness for the case when all players have an equal share in the estate:

Definition K: A player i may reasonably consider a set x of goods and dollars unfair if and only if $v_i(x) < C_i$ for some C_i satisfying the following two conditions:

1) There is at least one allocation $\underline{a}$ such that $v_i(a_j) \geq C_i$ for all j, and

2) For any allocation $\underline{a}$ there is at least one j such that $v_i(a_j) \geq C_i$.

This definition of fairness is in the spirit of divide and choose schemes. The first restriction on the C_i requires each player to be able to split the estate into n pieces (or, equivalently, devise an allocation $\underline{a}$) any one of which would be acceptable as a fair share of the estate. The second restriciton requires that for any division of the estate into n pieces, there is at least one piece that any particular player considers a fair and acceptable share of the estate.

For the case of two players, the restrictions on C_1 and C_2 assure that the divide and choose scheme results in an allocation considered fair by both players. Kuhn proves that both his own divide and choose scheme for more than two players and the scheme of Knaster and Banach result in allocations which are fair by definition K.

Definition K appears reasonable for dividing a homogeneous finely divisible commodity fairly. However, the definition is less satisfactory if the estate is not homogeneous. Consider the following example:

<u>Example I.3</u>: Three players are to divide three mittens and $24 equally. The three players have identical value functions; each considers single mittens worthless, two or three mittens worth $12, and i mittens together with d dollars worth d dollars plus the value of i mittens.

The second restriction in definition K requires each players to consider one mitten plus \$8 acceptable; simply consider the allocation $\underline{a}$ with all a_i equal to one mitten plus \$8. However, this allocation, though fair according to definition K, is not Pareto optimal since it is dominated by the allocation ((three mittens), \$12, \$12) in which each player receives \$12 worth of goods. It appears necessary to modify Kuhn's definition to require that the final allocation be Pareto optimal among the set of fair allocations. This paper will require that any allocation of goods be Pareto optimal, thus eliminating the above illustrated problem.

I.9 Extensions to Rational Shares

Inherent in definition K is that the players are to divide the estate equally. Dubins and Spanier [7] suggest an extension of definition K to the case in which all players have positive rational shares in the estate. If all f_i, the players' shares, are positive rational numbers then they may be written as $f_i = k_i/k$ where all the k_i are positive integers such that $\sum_{i=1}^{i=n} k_i = k$.

The extension of Dubins and Spanier is equivalent to giving each player i k_i agents who are to allocate the estate fairly among themselves according to definition K. Thus player i receives the union of the shares of the k_i agents and player i receives a fair share whenever each of the k_i agents receives a fair share according to definition K in the k player problem.

Here fairness is defined in terms of fair shares for each of the agents. When a player's value function is not additive in all items, some difficulty may arise from the players' agents acting independently. In

particular, it is difficult to assure Pareto optimality of the final allocation.

<u>Example I.4</u>: Two players, with shares of one third and two thirds, are to divide an estate consisting of four mittens and \$12. The players have identical value functions; each considers single mittens worthless, two or more mittens worth \$12, and i mittens together with d dollars worth d dollars plus the value of i mittens.

In this example, the second player will have two agents. It is assumed that each agent will behave according to the corresponding player's value functions. This the allocation (\$12, (two mittens), (two mittens)) is fair according to definition K, and is even Pareto optimal among such fair allocations. Depending on which agents correspond to which player, the second player will receive either four mittens (the \$12 goes to the first player) or two mittens together with \$12 (the remaining two mittens go to the first player).

Since the first player values two mittens at \$12 and the second player values two mittens together with \$12 in excess of \$12, the second alternative dominates the first. Thus requiring the agents to allocate the goods Pareto optimally among themselves is not sufficient to assure that the final allocation is Pareto optimal. It may be difficult to modify definition K so that the k_i agents of player i may define their fair shares in a manner which assures the Pareto optimality of the final allocation.

A different extension of definition K to rational shares without requiring any agents is the following:

Definition K': A player i with a share $f_i = k_i/k$ in the estate may reasonably consider a set x of goods and dollars unfair if and only if $v_i(x) < C_i$ for some number C_i satisfying the following two conditions:

1) There is at least one allocation $\underline{a} = (a_1,a_2,\ldots,a_k)$ of the goods among k imaginary agents such that $v_i(U_{j\epsilon J}\ a_j) \geq C_i$ for all subsets J of $\{1,2,\ldots,k\}$ cardinality k_i, and

2) For any allocation $\underline{a} = (a_1,a_2,\ldots,a_k)$ there is at least one set J of cardinality k_i such that $v_i(U_{j\epsilon J}\ a_j) \geq C_i$.

Again, it is reasonable to require that the final allocation is Pareto optimal. Even with this requirement, definition K' gives rise to some questionable allocations.

Example I.5: Two players, with shares of one third and two thirds, are to divide an estate of three mittens where the players have value functions as in examples I.3 and I.4.

Consider the symmetric allocation among the three imaginary agents where each a_i consists of one mitten and \$8. The first player must thus, by the second restriction in definition K', consider one mitten together with \$8 a fair and acceptable one third share of the estate, which is worth considerably less than one third of this player's value for the entire estate. Using this allocation among the imaginary agents would result in a final allocation of two mittens and \$16 to the second player and one mitten and \$8 to the first player.

Although it has never been precisely stated what a "share" in an estate means, the above final allocation does not seem to be a one third to

two third split. The problem is not one of Pareto optimality, since it may be verified that the allocation is indeed Pareto optimal, but one of whether the allocation is a one third to two third division of the estate.

If it is possible to convince oneself that the above does actually constitute a reasonable division, then one should consider the following example:

Example I.6: Two players are to divide an estate of three mittens and a debt of $12. The players' shares and value functions are as in the previous example.

The symmetric allocation among the three imaginary agents has each a_i equal to one mitten and a debt of $4. The resulting final allocation gives player one a mitten and a debt of $4 (and gives player two a pair of mittens and a debt of $8). It is hard not to sympathize with player one who upon receiving a one third share of an estate both players value at zero has to pay a net debt of $4. In this example, such division appear unreasonable.

As Kuhn mentions, his definition of fairness, although seemingly satisfactory for allocating homogeneous finely divisible items, is not necessarily as satisfactory for dividing an inhomogeneous or indivisible item; Kuhn suggests considering a cake with some indivisible cherries imbedded in it. A strength of the definition is that it does not require any consideration of "values" in terms of some extraneous commodity for problems already containing sufficient (how ever much that may be!) devisable goods. For some problems, such as example I.4, it is unlikely

the players will consider the items sufficiently divisible to avoid the introduction of some extraneous commodity; a commodity which may be used both to transfer "value" between players and used to evaluate players' relative preferences. This paper assumes the presence of a homogenous finely divisible transferable commodity ("dollars") and uses it in the alternate definitions of fairness considered in the next chapter.

I.10 The Role of "Dollars"

The choice of the distinguished finely divisible commodity has several ramifications. The mere presence of this, perhaps extraneous, commodity might tempt one to define fairness and fair shares in terms of it. Some care must however be exercised.

Example I.7: Two players, with shares of one third and two thirds, are to divide a homogeneous cake which has already been cut into nine equal size pieces which may not be divided further. Assume there is a general consensus among everyone, including the two players, that no cake is worth \$0, one third of the cake is worth \$2, two thirds is worth \$7, and a whole cake is worth \$9.

Example I.8: As in example I.6, but the players are to divide a pie, already in nine equal size pieces. Assume that no pie is worth \$0, one third of a pie is worth \$4, two thirds is worth \$5, and the whole pie is worth \$9.

In both of these examples the value functions are additive; the value of the entire cake or pie is the value of one third of the cake or pie

is the value of one third of the cake or pie plus the value of the remaining two thirds. Assigning three of the nine pieces of cake or pie to the first player and the remaining pieces to the second player results in a one third to two thirds division of the actual commodity.

These divisions do not result in a one third to two thirds division of the dollar value of the goods. The problem arises not from any lack of additivity of the value functions (they are additive), inhomogeneousness of the commodity (the goods are assumed to be homogeneous), or from the indivisibility of the nine original pieces; rather, the difficulty results from the lack of a common reference for defining "shares" in the estate. In this paper it is assumed that the shares are meant to be viewed with respect to the "dollar" value of the estate.

This assumption does not resolve all difficulties. Even in the case of equal shares, the determination of a fain allocation may depend on what commodity plays the role of dollars.

Example I.9: Two players are to divide equally an estate consisting of a single painting. Both players value the painting at $10, but value it at 10 tons and one ton of sand respectively. (The players value a ton of sand at $1 and $10 respectively).

If conventional money is used for the distinguished commodity, then the only fair allocation according to the above axioms is for one player to be awarded the painting, and for that player to pay $5 to the other player. This is a very reasonable allocation and might easily be the only fair allocation under any alternate fairness definitions which might be suggested.

If sand plays the role of dollars then there are many fair allocations of the estate. If the painting is assigned to the first player, then payment from the first to the second player of any amount between one half and five tons of sand results in a fair allocation according to definition K. If the amount of sand transferred is strictly between one half and five tons, then each player receives strictly more than one half of the estate (in terms of that player's value for the estate in sand). Unlike the allocations in terms of conventional dollars, it is possible for both players to receive more than their perceived share when computing in terms of sand.

The problem is not simply that the second player under-values the painting in terms of sand. It is quite possible that two individuals have different monetary values for sand. If both have the same monetary value for the painting, then in order to be consistent, they must have different values for the painting in terms of sand.

The ambiguity is in identifying what commodity should be considered the distinguished transferable commodity. Fortunately, in many real situations there is a natural choice. It will be assumed hereafter that there is indeed such a natural choice.

I.11 Fair Allocations of Dubins

The concepts of fairness considered to this point assume that a player's value for an allocation depends only on the set of goods that player receives. Dubins [6] suggests a scheme for fairly allocating an estate consisting solely of indivisible items (thus $D = 0$) among players with equal shares. In this scheme players may express their preferences as to how goods they do not receive are assigned among the remaining players.

Recall that an allocation $\underline{a} = (a_1,a_2,\ldots,a_n)$ is an assignment $\underline{s} = (s_1,s_2,\ldots,s_n)$ together with a vector $\underline{d} = (d_1,d_2,\ldots,d_n)$ whose components sum to zero. In particular, the allocation $\underline{a}$ may be written as $(s_1+d_1, s_2+d_2,\ldots,s_n+d_n)$. If the allocation $\underline{a}$ is one such that player i considers any a_j a fair share (in the spirit of the first restriction on C_i in definitions K and K'), then the number d_j may be interpreted to be the number of dollars player i must receive in addition to the set of goods s_i before player i considers the allocation a_i fair.

Consider all possible permutations of the coordinates of $\underline{a}$; when the a_j are distinct each permutation represents a different allocation. If $P: N \to N$ is the permutation function, then under the permutation P, player i will receive $a_{P(i)}$. It is now possible to interpret $d_{P(i)}$ as the amount player i must be paid before considering the permutation of the assignment $\underline{s}$ fair.

Note that since the d_j sum to zero, the sum of $d_{P(i)}$ over all possible perumtations P must also be zero for each player i. The scheme of Dubins may be derived by turning this observation around; let each player specify $d_{P(i)}$ for all permutations subject only to the restriction that for each player and each assignment of goods, the $d_{P(i)}$ summed over all permutations gives zero.

Modifying the notation slightly to reflect that $d_{P(i)}$ depends on both the permutation P and the assignment being considered, let $d_i(P,\underline{s})$ be the amount player i must be paid before considering receiving $s_{P(i)}$ and $d_i(P,\underline{s})$ and the remaining goods assigned according to $\underline{s}$ a fair and equitable allocation. Notice that player i has no direct influence on how much money the other players will receive.

Define two assignments to be distinct if neither is a permutation of the other. Then Dubins suggests the following allocation scheme. Let each player specify $d_i(P,\underline{s})$ subject only to the restriction that the sum of $d_i(P,\underline{s})$ over all permutations of all distinct assignments be zero for each player. Choose the permutation and assignment, say P^* and $\underline{s}^*$, which minimizes the sum $R(P,\underline{s}) = \sum_{i=1}^{i=n} d_i(P,\underline{s})$.

By the restriction that for each player, the $d_i(P,\underline{s})$ sum to zero, the sum over all permutations of all distinct assignments of $R(P,\underline{s})$ must also sum to zero. Thus, not all of the $R(P,\underline{s})$ may be positive; and therefore the minimum, $R(P^*,\underline{s}^*)$, must be at most zero.

Since $R(P^*,\underline{s}^*)$ is less than or equal to zero, it is possible to allocate each player i the set $a_i^* = s^*_{P^*(i)} + d_i(P^*,\underline{s}^*)$ and have an excess of $-R(P^*,\underline{s}^*)$ left over. This excess may be split among the players in any fashion desired; this discussion will assume that each player receives a non-negative of the excess. Thus, in the final allocation player i receives at least $s^*_{P^*(i)}$ and $\$d_i(P^*,\underline{s}^*)$, which is precisely the amount player i specified as fair when the remaining goods are assigned (as they actually are) according to the permutation P^* of the assignment $\underline{s}^*$.

The scheme of Dubins is best illustrated by an example.

Example I.10: Three players, with equal shares, are to divide an estate consisting of a single painting which they value at \$3, \$6, and \$9 respectively.

For this problem, with one item and three players, there are three distinct assignments; the painting may be assigned to any one of the three players.

Thus, for this scheme, each player must specify three values; $v_i(j)$ is the amount i must be paid if player j receives the painting, where by assumption $\sum_{j=1}^{j=3} v_i(j) = 0$ for each i.

Dubins suggests the following as "max-min" strategies or bids.

painting assigned to player j	=	1	2	3
$v_1(j)$	=	-2	1	1
$v_2(j)$	=	2	-4	2
$v_3(j)$	=	3	3	-6

The permutation with minimum sum corresponds to the right most column, and assigns the painting to the third player at a cost of \$6, and gives \$1 and \$2 respectively to the first and second players. If the excess of \$6 - (\$1 + \$2) = \$3 is split *proportionally*, then the resulting allocation is (\$2, \$3, Painting - \$5); precisely that of Knaster.

However, unlike in Knaster's scheme, the players may express preferences as to which of the "other" two players receives the painting. For example, the first player may prefer the second player receiving the painting over the third player receiving it. Thus, the first player may specify values of -2, .5, and 1.5 instead of the -2,1, and 1 displayed above. Now one of three possibilities (depending on the remaining players' bids) will occur: (i) the painting is awarded to player one at a cost not exceeding \$2, (ii) the painting is awarded to the second player and the first player receives at least \$.50, or (iii) the painting is awarded to the third player and the first player receives at least \$1.50. Thus the first player receives more money if the painting is awarded to the third player than when it is awarded to the second.

This scheme does not require that individual's value functions be additive in all goods for problems with more than one item; the players specify a "value" for each possible (distinct) assignment of the goods. However, the scheme does assume that the players have equal shares in the estate. This assumption seems difficult to avoid for this parituclar fair allocation scheme.

Even if the players all have non-zero rational shares in the estate, the use of agents (as in the first proposed extension of Kuhn's definition) is not satisfactory. The difficulties associated with agents in the modified form of Kuhn's definition of fairness will occur here. In particular, allocations similar to those arising in example I.4 may occur. Thus, the use of agents is not a satisfactory extension to unequal shares.

Other modifications might be considered. Possibly, players with large shares may be allowed to specify "values" which sum to something larger than zero, while players with small shares must restrict their sum to less than zero; presumably these limits on players' sums themselves still sum to zero so that there must be a non-negative excess. However, such a modification appears difficult to actually implement.

The scheme of Dubins allows players to indicate their preferences as to how goods they do not receive are assigned among the remaining players. This is possible by specifying different values of $d_i(P,\underline{s})$ for combinations of P and $\underline{s}$ which give player i the same set of goods $s_{P(i)}$. Although such flexibility is in general desirable, it requires the condition that the $d_i(P,\underline{s})$ sum to zero for each player; this gives an implicit, though perhaps obscure, interpersonal value comparison.

In actual fair division situations it may be difficult to convince the players that it is reasonable to require the above restriction on the $d_i(P,\underline{s})$. However, some restriction is clearly necessary...it would not do for me to insist on receiving at least five million dollars no matter who received the expensive painting which we are to divide! An alternate scheme, one in which players need not have equal shares, which permits players to express preferences about how the remaining goods are assigned will be presented later.

I.12 Summary

This chapter defined the problem being considered in some detail. The assumption, implicit in traditional auctions, that value functions are additive in all goods has several implications and drawbacks; a more general auction scheme, which avoids some of these difficulties, is considered. This more general scheme will be used throughout the paper.

Several "classical" fair division schemes are examined. The weaknesses of these schemes are illustrated by various examples. None of the classical schemes appear totally satisfactory. A considerable number of alternates will be consider in subsequent chapters.

CHAPTER II

CONCEPTS OF FAIRNESS

II.1 Introduction

The discussion and examples in chapter I indicate that the fairness definitions presented there are not totally satisfactory for the fair division problem under consideration. This chapter will present a number of alternative concepts of fairness and finally single out a particular definition for further study.

Two quite different view-points of fairness will be considered. Concepts which define fairness in terms of each player receiving the appropriate share of each item will be considered first. Alternatively, fairness may be defined by comparing the set of goods which one player receives to the set other players receive.

One particular fair allocation, which is closely related to several different allocations arising from the various definitions of fairness, will be singled out for further study. It is shown that for this allocation scheme the assumption that value functions are additive in dollars may be substantially relaxed without affecting the results.

The chapter concludes by considering schemes which allow players to express their preference on how the goods they do not receive are to be assigned among the remaining players. With appropriate restrictions on players' value functions, several of the fairness concepts considered above may be extended to this case.

II.2 "Individually-Reasonable" Allocations

In Knaster's fair allocation scheme each player i received at least

an f_i share of the preceived value of each good; that is, for any allocation $\underline{a}$ according to this scheme player i receives goods worth $v_i(a_i)$ and this is at least $\sum_{i=1}^{i=m} f_i\, v_i(j)$ where $j = 1,2,\ldots,m$ are the m indivisible goods. This sum is equal to f_i of the sum $\sum_{i=1}^{i=m} v_i(j)$ which, since Knaster implicitly assumes the value functions to be additive in all goods, is f_i of $v_i(M)$. Thus a fair share is at least f_i *of* the value player i places on the entire estate.

It is possible to use these observations to motivate a definition of fairness in situations where the value functions need not be additive in all goods.

Definition i-R: An allocation $\underline{a}$ is individually-reasonable if and only if $v_i(a_i) \geq C_i = f_i\, v_i(M+\$D)$ for all players i.

Merely stating a definition of fairness does not assure that there are any allocations which satisfy the definition; ideally there are Pareto optimal allocations which satisfy the fairness definition. It is shown below that there are Pareto optimal allocations which satisfy definition i-R.

Consider any revenue maximizing assignment $\underline{s}$ and the set of allocations given by $a_i = s_i - v_i(B_i) + f_i\, v_i(M+\$D) + r_i$ where the r_i are arbitrary real numbers such that their sum $\sum_{i=1}^{i=n} r_i$ is equal to the excess $E = R^* + D - \sum_{i=1}^{i=n} f_i\, v_i(M+\$D)$. It is easy to verify that such $\underline{a}$ are Pareto optimal allocations. Thus, if it is possible to choose all the r_i to be non-negative, then $v_i(a_i)$ is at least $f_i\, v_i(M+\$D)$ for all players i and the allocation $\underline{a}$ is individually-reasonable.

Lemma II.1: Let the excess $E(\underline{s})$ be defined by $E(\underline{s}) = R(\underline{s}) + \$D - \sum_{i=1}^{i=n} f_i\, v_i(M+\$D)$. If $\underline{s}^*$ is a revenue maximizing assignment, then $E(\underline{s}^*) \geq 0$.

Proof: Let i^* be a player with the maximum value for $v_i(M)$. Consider the assignment $\underline{s}$ where $s_{i^*} = M$ and $s_i = \emptyset$ for all $i \neq i^*$. Note that $v_{i^*}(M) = v_{i^*}(s_{i^*}) = R(\underline{s}) \leq R(\underline{s}^*) = R^*$. Now,

$$R(\underline{s}^*) + \$D - \sum_{i=1}^{i=n} f_i\, v_i(M+\$D)$$
$$= R(\underline{s}^*) + \$D - \sum_{i=1}^{i=n} f_i\, (v_i(M) + \$D)$$
$$= R(\underline{s}^*) - \sum_{i=1}^{i=n} f_i\, v_i(M)$$
$$\geq R(\underline{s}) - \sum_{i=1}^{i=n} f_i\, v_i(M)$$
$$\geq R(\underline{s}) - \sum_{i=1}^{i=n} f_i\, v_{i^*}(M)$$
$$= R(\underline{s}) - v_{i^*}(M) = 0 \text{ as desired.}$$

Thus for each revenue maximizing assignment $\underline{s}^*$ there is at least one Pareto optimal allocation which is an individually-reasonable allocation. In those instances when $E(\underline{s}^*) > 0$ there is an infinite collection of individually-reasonable allocations corresponding to the different sets of non-negative r_i which sum to $E(\underline{s}^*)$. By theorem I.1 all Pareto optimal allocations are obtained from revenue maximizing assignments. In order for the allocation $\underline{a}$ to be individually-reasonable, the number of dollars in a_i must be at least $f_i\, v_i(M+\$D) - v_i(s_i)$ where s_i is the set of goods other than dollars in a_i. This proves the following:

Lemma II.2: The individually-reasonable Pareto optimal allocations are precisely those allocations $\underline{a}$ with $a_i = (s_i - v_i(s_i) + f_i\, v_i(M) + r_i)$ where $\underline{s}$ is any revenue maximizing assignment and the r_i are any

non-negative real numbers such that $\sum_{i=1}^{i=n} r_i = E(\underline{s})$.

Unless all of the $v_i(M)$ are equal to R^*, the above lemma gives not a unique individually-reasonable Pareto optimal but an infinite collection of such corresponding to the different possible values of the r_i--the different ways the excess can be split among the players. One possible way to split the excess is to require $r_i = f_i\, E^*(\underline{s}^*)$, where $\underline{s}^*$ is any revenue maximizing assignment. In this case, the excess will be split in proportion to the players' shares.

This method for splitting the excess is the classic choice and is used by both Knaster and Dubins. There is also a natural motivation for this method of splitting the excess. Consider that the original estate is actually $M + \$D + E^*$. Since the value functions are assumed to be additive in dollars, the individually-reasonable Pareto optimal allocations will be those with $a_i = s_i - v_i(s_i) + f_i\, v_i(M+\$D+E^*)$, where $\underline{s}$ is any revenue maximizing assignment. If there is a unique revenue maximizing assignment then this allocation will be unique.

Splitting the excess of an individually-reasonable allocation proportionally is one natural possibility. Another possibility, to be discussed later, will result from splitting the excess differently; it is equivalent to splitting the excess from an alternate allocation scheme proportionally.

It is possible for an individually-reasonable allocation $\underline{a}$ to have a_i and a_j consist solely of dollars and $a_i > a_j$ even though $f_i < f_j$. Consider the individually-reasonable allocations for the following examples when the excess is split in proportion to the f_i.

Example II.1: (previously appeared as example I.9) Three players, with equal shares, are to divide an estate consisting of a single painting which they value at \$3, \$6, and \$9 respectively.

Example II.2: Three players, with shares of 1/2, 1/4, and 1/4, are to divide an estate consisting of a single painting which they value at \$2, \$96, and \$108 respectively.

In example II.1 the profit maximizing assignment awards the painting to the third player at a charge of \$9. The players then receive \$1, \$2, and \$3 respectively as their shares in the painting; leaving an excess E^* of $9 - (1 + 2 + 3)$, or three dollars. Using proportional division of the excess, each player receives one of these three dollars. This results in the final allocation of (\$2, \$3, (painting - \$5)).

This allocation has a disturbing aspect. Although the first two players have the same share in the estate, the second player is allocated 50% more than the first player. The first player may claim, based on the third player's bid, that there is an obvious market for the painting at \$9, and that it is unfair to be penalized for not being aware of this market. Arguing in this manner, both of the first two players would claim their bids should also be \$9 and that the resulting allocation should be (\$3, \$3, (painting - \$6)). Note that this is also an individually-reasonable allocation, but with a division of the excess not proportional to the shares, based on the players' original bids.

The second example is perhaps even more disturbing. The revenue maximizing assignment awards the painting to the third player at a cost of \$108. The players then receive \$1, \$24, and \$27 as their shares in

the estate, leaving an excess of 108 - (1 + 24 + 27), or $56. Thus the individually-reasonable allocation with the excess divided proportionally is ($29, $38, (painting - $67)).

The first player, with a one half share in the estate, receives only $29; less than the $38 allocated to the second player, with only a one fourth share. Again the first and second players may argue that they should both bid $108 and that the estate should be allocated according to ($54, $27, (painting - $81)). In this suggested allocation the first player receives exactly twice as much as the second player; the same ratio as their shares.

One further difficulty of basing a player's fair share on that player's perceived value of the entire estate is that it may be meaningless to ask the player to place a bid on the entire estate. If there is a large number of relatively poor individuals dividing a very large estate then it is likely that no player will want to be allocated a large fraction of the estate. Examples of such situations are corporate bond issues, or dividing responsibility for the welfare program.

In such cases, it can be assumed a priori that the bids for very large subsets of the estate will be relatively small and that the revenue maximizing assignment is extremely unlikely to award any one player a large set of goods. When the values of large sets enter into the problem only to the extent that they are assumed small enough not to be considered by the revenue maximizing assignment, then it appears unreasonable to base players' fair shares on their, presumably inaccurate, estimates of the value of the entire estate.

II.3 "Overall-Reasonable" Allocations

When it is difficult for individuals to place a value on the entire estate, one might consider defining a fair share in terms of the maximal value of the estate; fair shares may be defined in terms of the value of a profit maximizing assignment. For estates consisting of a single item, the profit maximizing assignment has the value of the highest bid and the suggested scheme corresponds to basing expectations on the highest bid for an item. It was noted in the discussion of examples II.1 and II.2 that this scheme would result in an individually-reasonable allocation with some, usually not proportional to shares, split of any excess.

Alternatively, players may desire their share f_i of their bid, or perhaps even of the maximum bid, of each item. However, if the value functions are subadditive (the value for large sets is significantly less than the sum of the values of the individual items in the set) it may be impossible for all such expectations to be satisfied. There is no assurance that the value of the revenue maximizing assignments is at least as large as any of the player's sum of values for single items.

A slightly weaker, but similar, concept is for each player to receive value equal to at least f_i of the revenue value for the sets of items different players are actually assigned. In other words, player i might expect $v_i(a_i) \geq \sum_{j=1}^{j=n} f_j(s_j)$ where s_j is the assignment, presumably revenue maximizing, actually used. Assuming that, in order to assure Pareto optimality of the allocation, only revenue maximizing assignments are considered, then this enables each player i to expect at least $f_i\ R^*$. Since the f_i sum to one, it is possible for each player to be allocated goods with $v_i(a_i) = f_i\ R^*$. Again, as in the previous discussion, players expect their share f_i of the maximum possible revenue.

If a reasonable share of the estate is defined in terms of the maximum individual player's value for the entire estate, rather than in terms of the corresponding player, then fair allocations are determined by definition o-R.

Definition o-R: An allocation $\underline{a}$ is overall-reasonable if and only if $v_i(a_i) \geq C_i = f_i \text{ maximum}_{j \in N} v_j(M+\$D)$ for all players i.

This definition is identical to that for individually-reasonable allocations except for the minimum fair share. If after the players have specified $v_i(M)$, all these values are increased to the maximum $v_i(M)$ then the individually-reasonable allocations with respect to these modified values are precisely the overall-reasonable allocations with respect to the original data.

Clearly, if $v_i(a_i)$ is at least f_i of the maximum $v_j(M)$ then it is at least f_i of $v_i(M)$. This verifies the following lemma.

Lemma II.3: Overall-reasonable allocations are also individually-reasonable.

As for individually-reasonable allocations, a corresponding result for excesses holds.

Lemma II.4: Let the excess $E(\underline{s})$ be defined by $E(\underline{s}) = R(\underline{s}) - \text{maximum}_{j \in N} v_j(M)$. If $\underline{s}^*$ is a revenue maximizing assignment, then $E(\underline{s}^*) \geq 0$.

Proof: Since a revenue maximizing assignment has at least the value of assigning all goods to any one player $R(\underline{s}^*) \geq \text{maximum}_{j \in N}\ v_j(M)$.

Lemma II.5: The overall-reasonable Pareto optimal allocations are precisely those allocations $\underline{a}$ with $a_i = (s_i - v_i(s_i) + f_i\ \text{maximum}_{j \in N}\ v_j(M) + f_i\ D + r_i)$ where $\underline{s}$ is any revenue maximizing assignment and the r_i are any non-negative real numbers such that $\sum_{i=1}^{i=n} r_i = E(\underline{s})$.

Unless the maximum $v_i(M)$ is equal to R^*, the above lemma gives an infinite collection of overall-reasonable Pareto optimal allocations corresponding to the different values of the r_i, i.e., the different ways the excess can be split.

Definition R: An allocation $\underline{a}$ is reasonable if and only if $v_i(a_i) \geq C_i = f_i(R^* + \$D)$ for all players i.

Lemma II.6: If $\underline{a}$ is a reasonable Pareto optimal allocation, then each $v_i(a_i) = f_i(R^* + \$D)$.

Proof: Any player receiving more value than f_i of $R^* + \$D$ results in the contradiction that the sum of players' values exceeds $R^* + \$D$.

Lemma II.7: The reasonable Pareto optimal allocations are precisely the overall-reasonable Pareto optimal allocations with any excess split in proportion to players' shares.

Proof: If $\underline{a}$ is an overall-reasonable allocation with proportional division of any excess, then for each player i, $v_i(a_i) = v_i(s_i - v_i(s_i) + f_i \text{ maximum}_{j \in N} v_j(M) + f_i D + f_i (R^* - \text{maximum}_{j \in N} v_j(M))$, which by cancelling terms and using the additivity of values in dollars reduces to $v_i(f_i R^* + f_i D)$ which is the desired $f_i(R^* + \$D)$.

The allocation scheme suggested in the opening comments of this section is to use reasonable Pareto optimal allocations. Since reasonable allocations are overall-reasonable, and they in turn individually reasonable, the reasonable allocations simply correspond to different ways of dividing any excess in the overall- and individually-reasonable allocations schemes.

The reasonable Pareto optimal allocation will be unique unless there is more than one revenue maximizing assignment. The fairness definitions have finally been restricted sufficiently to select an essentially unique Pareto optimal allocation as the equitable allocation. This particular allocation is both individually- and overall-reasonable. In addition, the next section shows that this allocation has additional properties associated with fairness.

II.4 "Proportional" Allocations

One motivation for reasonable allocations is examples II.1 and II.2 and the claim that players' ignorance of the estates market potential should not result in an inferior allocation to them. Adopting this philosophy, the definition of individually-reasonable allocations was restricted. An alternate modification gives rise to proportional allocations.

Recall that in the above examples, the individually-reasonable

allocation could allocate the sets $a_i = \$d_i$ and $a_j = \$d_j$ with $\$d_i > \$d_j$ even though $f_i \leq f_j$. An alternate is to consider the reasonable allocations. Notice that the reasonable allocations are more in the spirit of definition K'. Indeed, in the case of equal shares, the second condition of definition K' requires that for any division of the estate into n piles, the "best" (in the opinion of some player i) pile should be acceptable to player i as a fair share. Implicit in this argument is that each player might expect that $v_i(a_i) \geq v_i(a_j)$ for each of the remaining players j.

This comparison by one player i of $v_i(a_i)$ to the value of what each of the other players receive suggests an apparently quite different fairness definition.

<u>Definition P</u>: An allocation <u>a</u> is <u>proportional</u> if and only if $f_j\ v_i(a_i) \geq f_i\ v_i(a_j)$ for all players i and j.

It is not immediately clear from the definition that any proportional allocations exist. That such allocations do exist is shown by the following two lemmas. Indeed, in the case of equal shares, lemma II.7 shows that such allocations exist without requiring any additivity assumption on the value functions; all comparisions are made in terms of utilities.

<u>Lemma II.7</u>: If all f_i are equal (to 1/n) then there is at least one proportional allocation.

Proof: Let x_i be the value such that $u_i(x_i) = u_i(M + \$D - (n-1)\ x_i)$. Such a value must exist since the left hand side is a strictly increasing,

continuous, and unbounded function (by the assumptions on the utility functions) whereas the right hand side is strictly decreasing, continuous, and unbounded as a function of x_i. Let i^* be a player with maximal x_i. Then allocate the estate according to $a_{i^*} = (M + \$D - (n-1)\, x_{i^*})$ and $a_i = x_{i^*}$ for all $i \neq i^*$. This is a proportional allocation since

1. $(i \neq i^*)$ $u_i(a_i) = u_i(x_{i^*})$

$$\geq u_i(x_i)$$

$$= u_i(M + \$D - (n-1)\, x_i)$$

$$\geq u_i(M + \$D - (n-1)\, x_{i^*})$$

$$= u_i(a_{i^*})$$

implies that $v_i(a_i) \geq v_i(a_{i^*})$;

2. $(i,j \neq i^*)$ $a_i = a_j$ implies that $v_i(a_i) = v_i(a_j)$; and

3. $(j \neq i^*)$ $u_{i^*}(a_{i^*}) = u_{i^*}(M + \$D - (n-1)\, x_{i^*})$

$$= u_{i^*}(x_{i^*})$$

$$\geq u_{i^*}(x_j)$$

implies that $v_{i^*}(a_{i^*}) \geq v_{i^*}(a_j)$.

Note that the proof of lemma II.7 did not require any assumption about the value functions being additive in dollars. On the other hand, for unequal shares, and lemma II.8, such assumptions will be used.

Lemma II.8: There is at least one proportional allocation, and at least one which is also overall-reasonable.

Proof: Let $x_i = v_i(M + \$D)$ and let i^* be a player with maxiaml

x_i. Then allocate the estate according to $a_{i^*} = (M + \$D - (1-f_{i^*})\ x_{i^*})$ and $a_i = f_i\ x_{i^*}$ for all $i \neq i^*$. This is a proportional allocation since

1. $(i \neq i^*)\ x_{i^*} \geq x_i \Rightarrow v_i(x_{i^*}) \geq v_i(x_i)$

$$\Rightarrow f_{i^*}\ v_i(x_{i^*}) \geq v_i(x_i) + (f_{i^*}-1)\ v_i(x_{i^*})$$

$$\geq v_i(M + \$D - (1-f_{i^*})\ v_i(x_{i^*}))$$

$$\Rightarrow f_{i^*}\ v_i(f_i\ x_{i^*}) \geq f_i\ v_i(M + \$D - (1-f_{i^*})\ x_{i^*})$$

$$\Rightarrow f_{i^*}\ v_i(a_i) \geq f_i\ v_i(a_{i^*});$$

2. $(i,j \neq i^*)\ f_j\ v_i(a_i) = f_j\ v_i(f_i\ x_{i^*})$

$$= f_i\ v_i(f_j\ x_{i^*})$$

$$= f_i\ v_i(a_j);\text{ and}$$

3. $(j \neq i^*)\ f_j\ v_{i^*}(a_{i^*}) = f_j\ v_{i^*}(M + \$D - (1-f_{i^*})\ x_{i^*})$

$$= f_j\ (v_{i^*}(M + \$D) - x_{i^*} + f_{i^*}\ x_{i^*})$$

$$= f_j\ f_{i^*}\ x_{i^*}$$

$$= f_{i^*}\ v_{i^*}(f_j\ x_{i^*})$$

$$= f_{i^*}\ v_{i^*}(a_j).$$

This verifies that $\underline{a}$ is proportional. To verify that this allocation is also overall-reasonable, observe that

4. $(i \neq i^*)\ f_j\ v_i(a_i) \geq f_i\ v_i(a_j)$ for all players j

$$\Rightarrow f_{i^*}\ v_i(a_i) \geq f_i\ v_i(a_{i^*})$$

$$= f_i\ v_i(f_{i^*}\ x_{i^*})$$

$$= f_{i^*}\ f_i\ x_{i^*}$$

$$\Rightarrow v_i(a_i) \geq f_i\ x_{i^*}$$

$$= f_i\ \text{maximum}_{j \in N}\ v_j(M + \$D);\text{ and}$$

5. $(i \neq i^*)\ a_{i^*} = (M + \$D - (1-f_{i^*})\ x_{i^*})$

$$\Rightarrow\ v_{i^*}(a_{i^*}) = v_{i^*}(M + \$D - (1-f_{i^*})\ x_{i^*})$$

$$= v_{i^*}(M + \$D) - x_{i^*} + f_{i^*}\ x_{i^*}$$

$$= f_{i^*}\ x_{i^*} = f_{i^*}\text{maximum}_{j\in N}\ v_j(M + \$D).$$

Since there is a proportional allocation which is also overall-reasonable, there is some Pareto optimal proportional allocation (this is not necessarily a Pareto optimal allocation) which is also overall-reasonable.

Lemma II.9: At least one of the allocations Pareto optimal among the proportional allocations is also an overall-reasonable allocation.

Proof: If the allocation used in the proof of lemma II.8 is Pareto optimal among proportional allocations, then this is the desired allocation. If it is not Pareto optimal, then there is some allocation Pareto optimal among proportional allocations which dominates it. Since any allocation which dominates an overall-reasonable allocation is also overall-reasonable, this dominating allocation is the desired allocation.

The above lemma should not be interpreted as stating that there is a Pareto optimal allocation which is both proportional and overall- (or even individually-) reasonable. This conclusion is false; as illustrated by example II.3.

Example II.3: Two players with equal shares are to divide an estate

consisting of two candies, one red and the other green. The players' value functions are as below.

$x =$	$\emptyset$	R	G	$R \cup G$
$v_1(x) =$	0	5	7	9
$v_2(x) =$	0	2	5	9

For this example, it may be easily verified that the only allocation both proportional and individually-reasonable (which, in this example with $v_1(R \cup G) = v_2(R \cup G)$, is also overall-reasonable) is $((R \cup G - 4.5), 4.5)$. However, this allocation is dominated by the individually- (and overall-) reasonable allocation (R, G). Thus there need not be any Pareto optimal allocation which is both individually-reasonable and proportional.

That there are examples in which such Pareto optimal allocations do exist is illustrated by example II.4.

<u>Example II.4</u>: Two players with equal shares are to divide an estate consisting of two candies, one red and the other green. The players' value functions are as below.

$x =$	$\emptyset$	R	G	$R \cup G$
$v_1(x) =$	0	3	0	3
$v_2(x) =$	0	3	3	5

In this example, the allocations $((G - \$d), (R + \$d))$ with $0 \leq d \leq 3/2$ are the Pareto optimal proportional and individually-reasonable allocations. If $0 \leq d \leq 1/2$, then the allocations are also overall-reasonable; and if $d = 0$ then the allocation is reasonable.

II.5 "Fractionally-Proportional" Allocations

The lack of Pareto optimal allocations which are proportional suggests that definition P be relaxed. One possibility is definition f-P.

Definition f-P: An allocation $\underline{a}$ is f-proportional iff $f \cdot (f_j\ v_i(a_i)) \geq f_i\ v_i(a_j)$ for all players i and j.

Clearly, for sufficiently large f, any Pareto optimal allocation is f-proportional. Let f^* be (if one exists) the minimal f such that at least one Pareto optimal allocation is both f-proportional and individually-reasonable. In example II.3 the f^* is greater than one, whereas in example II.4 the f^* is less than one.

Indeed, in example II.4, f^* if equal to one half. This may be verified by considering the allocation ((R - 1), (G + 1)). Here
$1/2\ v_1(R - 1) = 1/2\ (3 - 1) = 1 = (0 + 1) = v_1(G + 1)$, and
$1/2\ v_2(G + 1) = 1/2\ (3 + 1) = 1/2\ (3 + 1) = 2 = (3 - 1) = v_2(R - 1)$.
Thus this allocation is the unique Pareto optimal allocation which is both f^*-proportional and individually-reasonable.

This allocation is not overall-reasonable, since d = 1 is not less than or equal to one half. It may be verified that the unique Pareto optimal allocation which is both f^*-proportional and overall-reasonable is ((R - 1/2), (G + 1/2)) with $f^* = 5/7$. This allocation is still not reasonable. Thus, in general, even with slightly modified definitions, the f^*-proportional allocations are not reasonable allocations. The nature of f^*-allocations will not be investigated further.

II.6 "Marginally-Proportional" Allocations

A slightly different version of proportional allocations is defined in terms of marginal values.

Definition m-P: An allocation $\underline{a}$ is marginally-proportional if and only if $f_j\, v_i(a_i) \geq f_i(v_i(a_i \cup a_j) - v_i(a_i))$ for all players i and j.

Roughly speaking, marginally proportional allocations assure that each player is awarded to least as much as the weighted marginal value of any other player's award. Although this concept is quite similar to, and coincides with if the values are additive, the concept of proportional allocations, the Pareto optimal reasonable allocations are always marginally-proportional.

Lemma II.10: The Pareto optimal reasonable allocations are also marginally-proportional.

Proof: By contradiction, assume there is a reasonable allocation $\underline{a}$ which is not marginally-proportional. Then, by lemma II.7, $a_i = s_i - v_i(s_i) + f_i R^*$ where $\underline{s}$ is the revenue maximizing assignment used to obtain $\underline{a}$. By assumption, there are players i and j such that

$$f_j\, f_i\, R^* = f_j\, v_i(a_i) < f_i\, (v_i(a_i \cup a_j) - v_i(a_i))$$

(note that this implies that $f_i \neq 0$)

$$= f_i\, (v_i(s_i \cup s_j - v_i(s_i) - v_j(s_j) + (f_i + f_j)R^*)$$

$$- v_i(s_i - v_i(s_i) + f_i R^*))$$

$$= f_i\, (v_i(s_i \cup s_j) - v_i(s_i) - v_j(s_j) + f_j\, R^*)$$

By subtracting $f_j\ f_i\ R^*$ from both sides and dividing the inequality by f_i,

$$0 < v_i(s_i \cup s_j) - v_i(s_i) - v_j(s_j), \text{ which implies that}$$

$$v_i(s_i \cup s_j) + v_j(\emptyset) > v_i(s_i) + v_j(s_j).$$

This contradicts the assumption that $\underline{a}$ was Pareto optimal, since $\underline{s}$ cannot be revenue maximizing assigning $s_i \cup s_j$ to player i and $\emptyset$ to player j results in an assignment with greater revenue than $\underline{s}$.

It is not in general true that all Pareto optimal marginally-proportional allocations are reasonable. It may be verified that all of the proportional allocations in example II.4 are also marginally-proportional. The only Pareto optimal reasonable allocation is (R, G). Thus, most of the marginally-proportional allocations are not the reasonable allocation in this example.

If all players' value functions are superadditive in all goods, then all marginally-proportional allocations are also proportional allocations.

<u>Lemma II.11</u>: If $v_i(a_i \cup b_i) \geq v_i(a_i) + v_i(b_i)$ for all players i and for all disjoint subsets a_i and b_i of the estate, then marginal-proportionality implies proportionality.

Proof: $f_j\ v_i(a_i) \geq f_i\ (v_i(a_i \cup a_j) - v_i(a_i))$ together with $v_i(a_i \cup a_j) \geq v_i(a_i) + v_i(a_j)$ imply $f_j\ v_i(a_i) \geq f_i\ v_i(a_j)$.

Thus, in the case when all value functions are superadditive, the fact that Pareto optimal reasonable allocations are marginally-proportional implies that these reasonable allocations are also proportional.

II.7 "Reasonable" Allocations for General Values

The above discussions indicate that the reasonable allocations have a wide variety of properties, and may be motivated by several seemingly different approaches. As will be shown later, the reasonable allocations also have desirable properties when the players view the fair division problem as a general game. Thus it seems appropriate to single out the reasonable allocations for further study.

This section examines the consequences of relaxing the assumption that value functions be additive in dollars. It is shown that under quite relaxed, and very plausible, conditions on the value functions that reasonable allocations may still be calculated. The generalized sealed bid auction is the special case of reasonable allocations where the auctioneer has a one hundred per cent share in the estate and all the bidders have a share of zero. Therefore, any results for reasonable allocations under the relaxed conditions also apply to the generlized auction.

In this section the value functions need not be additive in dollars. Since the value functions will be allowed to have different dependencies on dollars for different sets s, the notation will be changed to reflect this. Use $v_{i,s}(x)$ to represent what, in the previous notation, was $v_i(s \cup x)$. As before, it will be assumed that

1. $v_{i,\emptyset}(x) = x$ for all players i and all real values x.

In addition, assume that

2. $v_{i,s}(x)$ is a strictly increasing continuous function in x for each player i and each set s, and

3. $v_{i,s}(x)$ is an unbounded function, i.e. $\text{limit}_{x\to\infty(-\infty)}\, v_{i,s}(x) = \infty(-\infty)$ for each player i and each set s.

Notice that these assumptions are very plausible for actual situations. The first requires that the value of a set containing only dollars is precisely the amount of dollars in the set. The second requires that the value rise smoothly as the amount of money in the set increases. Finally, the third assumption requires that no matter how undesirable (or desirable) a subset containing zero dollars may be, the set becomes arbitrarily desirable (undesirable) if a sufficient number of dollars is added (subtracted).

For the remainder of this section, attention will be restricted to the case of reasonable allocations. For each player i and each set s of goods, define the bid function $b_{i,s}(R)$ as the bid player i would place on the set s knowing that the total revenue would be \$R, and thus knowing that $f_i R$ will be returned to player i. More precisely, $b_{i,s}(R)$ is to be the function for which $v_{i,s}(f_i R - b_{i,s}(R))$ (which is by definition $v_i(f_i R \cup s - b_{i,s}(R)))$ is equal to $f_i R$ for all R. The $b_{i,s}(R)$ may be interpreted as the bid that would be placed if player i had a total wealth of $f_i R$ in excess of the current situation. It should not be surprising that bids may depend on an individual's wealth.

Since the assumptions on $v_{i,s}$ imply that a continuous inverse function $v_{i,s}^{-1}$ exists, the above expression for $b_{i,s}$ may be solved for the bid function, with the result that

<u>Lemma II.12</u>: $b_{i,s}(R) = f_i R - v_{i,s}^{-1}(f_i R)$.

Since the inverse value functions are continuous, the bid functions are also continuous.

Finally, a fourth (implicit) assumption on the value functions is required. In particular, the value functions are assumed to be such that

4. $b_{i,s}(R)$ is a bounded function (above and below) for each player i and each set s. If some of the goods are divisible (thus permitting infinitely many subsets s), then also assume that the bound is uniform for all s.

This assumption, intuitively, states that no matter how wealthy an individual is, there is a bound on the value for any particular set s of goods.

With these assumptions it is still possible to find reasonable allocations.

<u>Theorem II.1</u>: If the value functions satisfy assumptions 1. to 4. above, then there is at least one value of R such that the total revenue resulting from a revenue maximizing assignment according to the bids $b_{i,s}(R^*)$ is precisely R^*.

Proof: Let $T(R)$ denote the revenue resulting from a revenue maximizing assignment based on the bids $b_{i,s}(R)$. It may be verified that $T(R)$ is a continuous function of R. However, by assumption 4., this sum is bounded above and below as R goes to plus and minus infinity. Thus for sufficiently negative R, $T(R) \geq R$. For sufficiently large R, $T(R) \leq R$. Therefore, since both $T(R)$ and R are continuous functions of R, there must be some intermediate value R^* such that $T(R^*) = R^*$.

The theorem shows that for reasonable allocations there exists at least one value of R such that if each player receives $f_i R$, and the bids are made accordingly, the resulting revenue will be precisely R. However there is no assurance that this value of R is unique. Indeed, if the players' value functions $v_{i,s}$ increase both more and less rapidly than linear (that is, if there are values x and y such that $d/dx\ v_{i,s}(x) > 1$ and $d/dx\ v_{i,s}(y) < 1$) then $b_{i,s}(R)$ will not be monotonic. Thus $T(R)$ need not be monotonic and may therefore be equal to R for several different values of R.

Although the possible non-uniqueness may be disturbing, the theorem shows that reasonable allocations still exist when working with the relaxed assumptions on the value functions. Thus, the assumption that values are additive in dollars simplifies the analysis, but does not change the existence of reasonable allocations. For simplicity, most of the subsequent analysis will be in terms of value functions additive in dollars. However, most of the results hold, with at most minor modifications, under the more relaxed assumptions.

II.8 Preferences Over Assignments

It is possible to extend fair allocation schemes to allow individuals to express preferences as to how the items in the estate are to be assigned among all of the players. A player's values may depend on how the goods are assigned among the remaining players. In particular, rather than specifying $v_i(s)$ for all subsets s of the estate, let the players specify $v_i(\underline{s})$ for all possible assignments $\underline{s}$, where it is assumed that $v_i(\underline{a})$ is equal to $v_i(\underline{s}) + d_i$ where $a_i = s_i + d_i$. Choose any assignment $\underline{s}$ which maximizes $R^* = \sum_{i=1}^{i=n} v_i(\underline{s})$; where $\underline{a}$ ranges over all

possible assignments of the goods (excluding any dollars) in the estate. Then a reasonable allocation would be $\underline{a}$ with $a_i = s_i - v_i(\underline{s}) + f_i R^*$.

Unfortunately, without some further restrictions on the value functions, the above defined extended fair allocation schemes are not operational--any player i may acquire a more valuable award by subtracting a positive number from each $v_i(\underline{s})$. Thus, some normalization will be necessary.

One possible normalization is that used by Dubins [6], to wit: the sum over all possible assignments of $v_i(\underline{s})$ must equal zero. Perhaps a more natural normalization is to require $v_i(\underline{s})$ to be at least zero whenever the assignment $\underline{s}$ assigns all m goods to a single player. This normalization indicates clearly the fact that any such normalization will imply some form of interpersonal utility comparisons. The problem of interpersonal utility comparisons is avoided in the original fair allocation scheme by requiring $v_i(\emptyset) = 0$ (even though this may also contain an interpersonal utility comparison).

Depending on the problem, there may be a natural normalization on the $v_i(\underline{s})$, and then the extended fair allocation scheme will enable players to express preferences as to how the goods in the estate are assigned to the players. Note that to motivate this extension, it was assumed that $v_i(\underline{s} + \underline{d}) = v_i(\underline{s}) + d_i$. However, the allocation scheme may be used without any such motivation, and this restriction of the v_i may then be unnecessary.

One model, which might be appropriate and reasonable in the case where an inherited estate is to be divided, or in the case of an auction, is to assume that in addition to the n players there is an additional

player called the auctioneer. Assume that the n ordinary players will not be allowed to resell or trade any commodities assigned to them (with the possible exception of money, which after all is hard to distinguish from any money not originally in the estate).

The auctioneer is not subjected to the above restriction on reselling items. It is possible to require that the auctioneer may not resell any good to the n original players; or alternatively it is possible to assume that the auctioneer may resell to anyone. (The players' bids may be different under these two possible restrictions on the auctioneer.)

For such a model it may be quite natural to require that $v_i(\underline{a})$ is at least zero whenever the entire estate is sold to the auctioneer. Although such a requirement involves an interpersonal comparison of utilities (based on the situation in which the auctioneer is assigned all the goods), this comparison may be acceptable.

In the case of an inherited estate, and especially in the case of an auction, the status quo is the situation where someone else (the auctioneer) has all the goods. One may argue that all utility or value measurements should be made from this status quo.

Perhaps the model above is not appropriate, or the requirement that $v_i(\underline{s})$ be non-negative for any assignment awarding all goods to the auctioneer may not be acceptable. Using the extended fair allocation scheme, it should be possible to restrict the value functions in such a way that it is clear what implicit interpersonal utility comparisons are made. Unlike the scheme of Dubins, in which the interpersonal comparison of utilities is somewhat obscure, the proposed extended fair allocation scheme should be adaptable to whatever comparison assumption is most natural in any particular instance.

II.9 Summary

Several different definitions of fairness have been examined. Some of them are based on defining a fair share in terms of players' values for the entire estate. Others are defined in terms of comparing what one player is awarded to what other players are awarded.

Despite the seemingly different approaches, these various definitions of fairness have one basic result in common. The reasonable allocations are a special case of many of the definitions; thus making them a particularly strong choice for further study. Most of the remaining chapters concentrate on reasonable allocations; fortunately most of the results easily extend to arbitrary overall- and individually-reasonable allocations.

CHAPTER III

AUCTIONS

III.1 Introduction

Theorem I.1 shows that there are two parts to the fair allocation problem. The first, discussed previously, is a consideration of how to choose from among the Pareto optimal allocations. The second part is the question of obtaining a Pareto optimal allocation.

The theorem indicates that one method of obtaining a Pareto optimal allocation is to first auction the estate according to a revenue maximizing assignment, and then divide the resulting revenue among the players according to the fairness definition appropriate in the particular situation. The reasonable allocation scheme for general value functions also depends heavily on identifying revenue maximizing assignments. Thus, auctions play a crucial role in determining Pareto optimal allocations.

Alternatively, if one player has a share of one in the estate, then the individually- and overall-reasonable allocation schemes (with any excess divided in proportion to player's shares) and the reasonable allocation scheme (using either the general or more restrictive assumptions on value functions) are simply auctions. The player with a one hundred percent share in the estate receives all of the revenue. (One indication of the general interest in auctions is that a recent bibliography [24] lists over 350 papers studying some aspect of auctions.)

The usual sealed bid auction, with reservation prices, in which each item is sold to a high bidder at the high price is the special case of the generalized auction where all value functions are additive in all commodities. This additivity assumption implicit in traditional auctions

was shown to be a very strong restriction and to have several undesired implications.

The implications of requiring value functions to be additive in all commodities resulted in a consideration of less restrictive assumptions on the value functions. Allowing players to bid on all possible subsets of the estate complicates matters greatly. Not only must players now specify a very large number of bids (or a large number of bid functions if some goods are divisible), but solving the set partitioning problem required in determining a revenue maximizing assignment is a very difficult mathematical problem [1].

One possible relaxation of the additive values assumed in the usual sealed bid auction is for the goods to be auctioned sequentially (an example of which will be given shortly). For each good in turn, there is a sealed bid auction in which players, *presumably*, bid their marginal values for the item based on any goods they may have already been assigned. Such a scheme greatly reduces the number of bids each bidder must prepare.

Such a scheme has two serious drawbacks. If a large number of items is to be auctioned, then there will have to be many sequential one item auctions; a very awkward procedure. Alternatively, the sequential auction scheme may be viewed as only a heuristic for solving the set partitioning problem; each player submits bids on all possible subsets and the auctioneer pretends that the auction is conducted sequentially.

Whether the sequential auction is viewed as an actual procedure or merely as a heuristic, one drawback remains. The assignment of goods resulting from such an auction need not be a revenue maximizing assignment.

This dilemma is not a result of choosing the "wrong" order to auction the items. Indeed, this chapter exhibits an example in which, regardless of the order the items are auctioned in, the sequential auction can not result in a revenue more than $1/n + e$ (where e is a arbitrary small positive real number) of the revenue maximizing assignment. This is an unsatisfactorily small fraction of the optimum for any sizable number n of players.

For problems with a small number of indivisible goods to be auctioned among a reasonable number of players the set partitioning problem may be solved exactly. An exact solution using dynamic programming requires on the order of $n\,3^m$ elementary mathematical operations. Although this enables a computer to solve problems for small values of n and m, the complexity of the problem grows enormously as the number m of goods increases.

The difficulty, or impossibility, of solving the general auction exactly suggests trying some heuristic. Hopefully, the heuristic will always obtain an assignment which is close to the revenue maximizing assignment in value. One common heuristic used to obtain approximate solutions to integer programs is the "greedy" heuristic.

The greedy heuristic is closely related to the sequential auction scheme, only it employs an order for assigning the items which is implicitly determined by the player's bids. There are problems for which the greedy heuristic does very poorly. Fisher, Nemhauser, and Wolsey show that the heuristic performs quite poorly under some quite restrictive assumptions about the additivity of the value functions. These results are extended to a wider class of problems, including all auctions with subadditive value functions. In particular, a "tight" bound on the

efficiency of the greedy heuristic for such problems is $1/m$ of the value of the revenue maximizing assignment.

A revenue equal to only $1/m$ of the possible revenue is very little indeed, especially if the greedy heuristic is considered only for problems with too many goods to be solved by dynamic programming. An example based on an auction that was actually conducted in the real world is used to establish this bound; thus the negative result is not a mere mathematical pathology. The chapter concludes with a "tight" bound, in terms of a characterization of the degree to which the value functions are additive, for arbitrary value functions.

This chapter concludes by considering a restriction on the value functions which results in an auction which is relatively easy to solve exactly using dynamic programming. In particular, if each player's function is solely a function of the number of items in a subset, then the crucial variable in the dynamic programming iterations is the number of rather than which particular items assigned to each player. This model may be applicable in situations where the items are roughly similar.

A similar result is obtained for the case in which there are several classes of similar items. Although the work required to solve the problem grows exponentially with the number of different classes, it is still quite reasonable for problems with three classes each consisting of ten or twenty items. Thus, although the chapter presents several discouraging results, there are situations for which the general auction scheme is feasible.

III.2 Sequential Auctions

One possible generalization (for estates consisting solely of m

indivisible goods) of Knaster's scheme is based on modifying the sealed bid auction to a sequential sealed bid auction. Consider an auction where the items are brought up for sale one at a time. For each item, players submit a sealed bid based on their marginal value for the item, and the item is sold to any player with the high bid. If all value functions are additive in all goods, then this sequential auction will result in the same assignment of goods as the usual sealed bid auction. If, however, not all the value functions are additive, the results may differ. In particular, the sequential auction will result in an allocation which accurately reflects the bidder's values for whatever set of goods is assigned.

Consider the following example.

Example III.1: The estate to be auctioned consists of two indivisible goods (R and G) which the auctioneer must sell (the auctioneer's reservation prices are arbitrarily small). There are two bidders; their value functions are as follows:

$x =$	$\emptyset$	R	G	$R \cup G$
$v_1(x) =$	0	5	7	9
$v_2(x) =$	0	2	5	9

In the usual sealed bid auction (where players are assumed to bid their values for individual items), both items would be awarded to the first player at a total cost of \$12. Unfortunately, \$12 is considerably larger than the players' value of \$9 for the set $R \cup G$.

In the sequential auction, there are two cases. If R is auctioned first, it will be sold to the first player. Then G will be sold to the second player since $v_2(G) - v_2(\emptyset) = 5$ is greater than $v_1(R \cup G) - v_1(R) = 2$. Thus the resulting assignment awards R to the first player and G to the second player. The total revenue is $10. In this scheme, each player pays a "fair" price for their awards.

If G is auctioned first, it will be sold to the first player. Now R may be sold to either player since both players have the same marginal value of $2 for R. The resulting revenue is $9. Although each player still pays a "fair" price for their award (either $R \cup G$ to player one, or G to player one and R to player two), this case results in less revenue than the $10 of the previous case. This the total revenue generated in an auction may depend on the order items are auctioned; this dependence is not unique to this particular type of sequential auction, but has been observed before [21] in other sequential auctions.

The example illustrates that some orders for auctioning the items produce more revenue than others, and there is an order which results in a revenue maximizing assignment. However, this is not always the case. Reconsider the following (previously example I.1).

Example III.2: Three indivisible items are to be sold among two players, whose values are indicated below.

$x =$	$\emptyset$	A	B	C	$A \cup B$	$A \cup C$	$B \cup C$	$A \cup B \cup C$
$v_1(x) =$	0	10	11	12	17	18	22	28
$v_2(x) =$	0	9	12	13	18	19	20	28

It may be verified that the unique revenue maximizing assignment awards $B \cup C$ to the first player and A to the second player. This assignment can not be achieved by any ordering of the goods in a sequential auction; if A is sold first, it will be awarded to the first player, whereas if B or C is sold first, B or C will be awarded to the second player. The first item auctioned will always be awarded to the "wrong" player. The value of the revenue maximizing assignment is \$31; any order of items in a sequential auction will result in a revenue of only \$30.

The above example is not so contrived so as to be unusual in real world situations. Indeed, the two players have very similar values for all sets of items, they value sets in the same relative order, and both players' value functions are slightly subadditive. The example, with appropriate units of value, might realistically illustrate two oil companies' bids for three off shore oil drilling leases. In such a situation, the inefficiency of the sequential auction might mean millions of dollars lost to the auctioneer.

The example shows that regardless of order, the sequential auction may be inefficient--the auction need not result in a Pareto optimal allocation. Although sequential auctions give close to the optimal revenue in the above example, such auctions may do much worse. The example below will indicate what poor assignments sequential auctions may generate.

Before considering the example, certain assumptions used throughout the remainder of this chapter should be stated explicitly. It is assumed that there is more than one player bidding for any items, and that there is more than one item to be sold. If there is one player or only one item, the "auction" is trivial. Such auctions may safely be ignored.

Example III.3: Let there be $n > 1$ players bidding for an estate M of $m > 1$ indivisible goods. Partition the estate into n sets $S_1, S_2, \ldots, S_n$ in any manner which makes the cardinalities of the sets as equal as possible. Thus, if $\mathrm{Int}(x)$ is the integer part of x and $\#(S_i)$ is the cardinality of S_i, $\mathrm{Int}(m/n) \leq \#(S_i) < \mathrm{Int}(m/n) + 1$.

Let k^* denote the maximum $\#(S_i)$, and let the value functions be as follows:

$$v_i(s_i) = 1 + e\ \#((M \backslash S_i) \cap s_i) \quad \text{if} \quad S_i \cap s_i = \emptyset; \text{ and}$$

$$v_i(s_i) = e\ \#((M \backslash S_i) \cap s_i) + \#(S_i \cap s_i) \quad \text{if} \quad S_i \cap s_i \neq \emptyset,$$

where $0 < e < 1$ and $(M \backslash S_i)$ denotes the set of items not in S_i.

The sets S_i may be interpreted as "player i's" goods; player i is willing to pay one unit for each item in S_i. The player is willing to pay $1 + e$ units for any single item not in S_i, but additional items not in S_i are worth only e units. It may be verified that the revenue maximizing assignment awards each player i the set S_i. The resulting revenue is \$m.

Consider now the performance of a sequential auction. Regardless of the order in which the goods are auctioned, the first $n-1$ items will be sold to "wrong" players. More precisely, for the first $n-1$ items sold, if an item is in S_i, it will not be sold to player i. Notice that now, as soon as some player i is sold a good from the corresponding S_i, then this same player will receive all the remaining goods not yet auctioned.

Thus the best possible resulting assignment has $S_i \subseteq s_i$ for some

one player i, and $S_i \cap s_i = \emptyset$ for all remaining players. This implies that the maximum revenue results when the n^{th} good to be auctioned is sold to the "correct" player (and even this can occur only if there are more than two players). Then all the remaining items are sold to this same player.

<u>Lemma III.1</u>: Using the sequential auction on example III.1 results (for an appropriate choice of e) in revenue less than $(n + m/n)/m$ of the maximum possible.

Proof: The above discussion indicates that the sequential auction can do no better than $n-1$ players each receiving an item not in the corresponding S_i, and the remaining player receiving all the remaining items, possibly (for $n > 2$) receiving all the items in the corresponding S_i. Thus at most k^* items will be sold to the "correct" player. The resulting revenue is at most $(n-1)(1+e) + k^* + e(m-k^*)$. But, $k^* < \text{Int}(m/n) + 1 \leq 1 + m/n$ implies that $n - 1 + k^* < n + m/n$. Thus for $e < ((m/n) + 1 - k^*)/(n - 1 + m - k^*)$, the resulting revenue is less than $n + m/n$. However, the revenue maximizing assignment results in a revenue of m; the ratio is therefore less than $(n + m/n)/m$.

The ratio of the maximum revenue from a sequential auction to the optimal assignment is $(n + m/n)/m = n/m + 1/n$. If m is much larger than n (there are many more goods than players), this tends to $1/n$; a rather small fraction of the optimal. For m not much larger than n, it is possible to consider similar examples in which $n^* < n$ of the players have value functions as in the example and the remaining players are dummies (their value functions are identically equal to zero). The resulting ratio is then $n^*/m + 1/n^*$. Since any n^* satisfying

$1 < n^* \leq n$ is allowable, the ratio may be reduced, as recorded in lemma III.2.

Lemma III.2: Using the sequential auction on example III.1 results (for appropriate e) in revenue less than $\text{minimum}_{\{n^*: 1 < n^* \leq n,\ n^* \text{ integer}\}}(n^*/m + 1/n^*)$ of that from the revenue maximizing assignment.

For m larger than four, this ratio is less than one. It is substantially less than one for m more than ten. It can not be expected that sequential auctions will in general efficiently assign goods if there are more than a very few items.

The revenue resulting from a sequential auction is very sensitive to the actual bids.

Example III.4: As in example III.3, but now let
$v_i(s_i) = 1 + e\ \#((M \backslash S_i) \cap s_i) - d$ if $S_i \cap s_i = \emptyset$, where $e < d < 1$.

If (as required in the proof of lemma III.1) e is very small, then d may also be very small and the difference between the value functions in examples III.3 and III.4 is very little. However, regardless of the order in which items are auctioned, using the sequential auction in example III.4 always results in a revenue maximizing assignment. Indeed, the resulting assignment is always the same; $s_i = S_i$ for all players. The resulting revenue is m.

A small change in the value functions may have a drastic effect on the efficiency of the sequential auction. In many actual situations,

bidders can not estimate their true values exactly. Thus, even in some situations in which bidders' true values are such that the sequential auction is efficient, any error in specifying the value functions may result in very inefficient assignments.

III.3 Exact Solutions

When there are only a few goods, all indivisible, it is relatively simple to compute the optimal assignment using dynamic programming. (For a general reference on dynamic programming, see [16].) The computation requires on the order of $n\ 3^m$ elementary arithmetic operations.

Lemma III.3: The revenue maximizing assignment can be calculated in $n\ 3^m$ elementary arithmetic operations.

Proof: The assignment is calculated using dynamic programming. In particular, arrange the players in some order; the proof uses the order $1,2,\dots,n$. Start the recursion by setting $N_0 = \emptyset$. At the i^{th} iteration, compute the optimal assignment of each subset of the estate among the players in $N_i = N_{i-1} \cup i$. For each subset, this requires determining what part of the subset should be assigned to the players in N_{i-1} according to the optimal assignment calculated in the previous iteration. Repeat this procedure until all players have been included.

Each step within an iteration involves considering all possible partitions of a given subset into two pieces S_1 and S_2. For each partition, compute the revenue resulting from assigning S_1 to player i and assigning S_2 optimally (according to the results calculated in the previous iteration) among the players in N_{i-1}. The partition with the maximum revenue indicates the optimal way of assigning the subset among

the players in N_i.

Each iteration involves one arithmetical comparison for each partition of the subset into two pieces. Exploiting the simple one to one correspondence between m digit ternary numbers and the possible partitions of subsets (the i^{th} digit is zero if item i is not in the subset, one if the item is in S_1, and two if the item is in S_2), there will be 3^m such comparisons. The dynamic program thus requires $n3^m$ comparisons. (In addition, at each iteration, the optimal assignment of 2^n subsets must be recorded.)

The lemma indicates that solving a problem of assigning ten items among five players will require about 300,000 units of computer time (where each unit is the time required to do one simple for a modern computer).

Increasing the number of items from ten to fifteen increases the amount of computer time required by a factor of 243, thereby adding minutes of computing time. For twenty items, a computer would require over a day of time; this is generally considered an extremely large amount of time.

III.4 Performance of the Greedy Heuristic

For fewer than ten items, the dynamic programming approach can give a revenue maximizing assignment in a reasonable amount of computer time. Dynamic programming is infeasible for much larger problems, and sequential auctions may do very poorly for such problems. This suggests using some heuristic for solving the assignment problem.

A commonly applied heuristic for solving integer programs is the "greedy" algorithm. Starting with all variables equal to zero, each

iteration of the algorithm increases (or, later, decreases) by one the variable which most increases the value of the objective function while maintaining feasibility of the solution. The greedy solution is any solution obtained by repeating this procedure until no further improvement of the objective function is possible.

This section considers a class of zero-one integer programs which arise from the question of optimally assigning $m > 1$ indivisible items among $n > 1$ players. Given the value $v_i(s_i)$ for each individual i and each subset s_i of the estate, the object is to find the revenue maximizing assignment.

Let X^i be an m component vector, and let $X = (X^1, X^2, \ldots, X^n)$ be the composite $m \cdot n$ component vector. The X^i_j are zero-one variables with $X^i_j = 1$ if and only if item j is assigned to player i. Thus, s_i is the set of items j for which $X^i_j = 1$. The problem may now be written as follows.

<u>Problem III.1</u>: Maximize $V(X) = \sum_{i=1}^{i=n} v_i(X^i)$ subject to $\sum_{i=1}^{i=n} X^i_j \leq 1$ for all j, and all $X^i_j = 0$ or 1.

The inequality constraint forces each item to be assigned at most once; for infeasible solutions, the s_i need not be disjoint. Applying the greedy algorithm to solve this problem is similar to the sequential auction scheme except for the imposed order. It is thus not surprising that the greedy algorithm can, in general, do very poorly.

Since the results hold for a wider class of integer programming problems than the above problem, a more general form will be considered.

Most of the examples, however, will be from the context of assigning items among players. In particular, consider the following problem.

Problem III.2: Maximize $V(X)$ subject to any constraints such that

1) all $X_j^i = 0$ or 1;
2) any X with exactly one $X_j^i = 1$ is feasible; and
3) feasible X have at most m components $X_j^i = 1$.

It is clear that problem III.1 is a special case of problem III.2.

Several restrictions will be placed on the form of the objective function. Some conditions which will be considered are the following:

1) Normality: $V(0) = 0$;

2) Monotonicity: $V(X) \geq V(Y)$ whenever $X \geq Y$;

3) Submodularity: $V(X) + V(Y) \geq V(X \cup Y) + V(X \cap Y)$ for all vectors X and Y;

3') Subadditivity: $V(X) + V(Y) \geq V(X \cup Y)$ for all vectors X and Y;

3") Discounted: $V(X) \leq \sum_{i,j:X_j^i=1} V(e_j^i)$ (where e_j^i is the unit vector with component i,j equal to one and all other components equal to zero) for all vectors X; and

3*) Variably Discounted: $V(X) \leq D(\#\{(i,j): X_j^i=1\}) \sum_{i,j:\ X_j^i=1} V(e_j^i)$ (where $\#(S)$ is the cardinality of the set S and D is a non-negative "discount" function) for all vectors X.

Alternatively, if $V(X) = \sum_{i=1}^{i=n} v_i(X_i)$, then the conditions may be stated in terms of the individual v_i. For concreteness, the conditions are

stated below.

$\underline{1}$) Normality: $v_i(\emptyset) = 0$ for all i;

$\underline{2}$) Monotonicity: $v_i(s_i) \geq v_i(t_i)$ whenever $s_i \geq t_i$ and for all i;

$\underline{3}$) Submodularity: $v_i(s_i) + v_i(t_i) \geq v_i(s_i \cup t_i) + v_i(s_i \cap t_i)$ for all sets s_i and t_i and for all i;

$\underline{3}$') Subadditivity: $v_i(s_i) + v_i(t_i) \geq v_i(s_i \cup t_i)$ for all sets s_i and t_i and all i;

$\underline{3}$") Discounted: $v_i(s_i) \leq \sum_{j:\ j \in s_i} v_i(j)$ for all sets s_i and all i; and

$\underline{3}$*) Variably Discounted: $v_i(s_i) \leq D_i(\#(s_i)) \sum_{j:\ j \in s_i} v_i(j)$.

There is a close connection between the two forms of the conditions. Some of these connections are listed in the following lemma.

<u>Lemma III.4</u>: If $V(X) = \sum_{i=1}^{i=n} v_i(X^i)$, then

1) Any condition $\underline{i}$ implies the corresponding condition i;

2) Conditions $\underline{1}$ and 2 together imply condition $\underline{2}$;

3) Condition $\underline{1}$ and any form of condition 3 together imply the corresponding form of condition $\underline{3}$. In the case of $\underline{3}$*, the D_i may all be equal to D.

Proof: The first claim is obvious. The remaining two may be verified by considering vectors X with all but one of the subvectors X^i identically equal to zero.

The conditions 3,3',3", and 3* have a similar character; they all

specify how non-additive the value functions may be. If all the value functions are additive in all goods, then the greedy algorithm is identical with the usual sealed bid auction and results in a revenue maximizing assignment. Fisher, Nemhauser, and Wolsey [9,17 & 18] study the performance of the greedy heuristic when the objective function satisfies conditions 1,2, and 3. In this section, similar results are obtained for the case when condition 3 is relaxed to one of the other forms 3" or 3*.

It is hinted above that the forms of condition 3 are listed in order of increasing generality; this will be verified. The "discounted" restriction is the special case of the "variably discounted" condition when the discount function is identically one. It may easily be verified that normality and monotonicity together with submodularity imply subadditivity. Likewise, normality and monotonicity together with subadditivity imply discountedness. The reverse implications may be shown to be false by considering appropriate examples.

Example III.5: Let $m = 3$, and let v_1 be given by

$x =$	$\emptyset$	A	B	C	$A \cup B$	$A \cup C$	$B \cup C$	$A \cup B \cup C$
$v_1(x) =$	0	2	2	2	3	3	3	5

and let $v_i(x) = 0$ for all x when $i > 1$.

It may be verified that $V(X)(= v_1)$ in example III.5 is normal, monotonic, and subadditive. However, if $s_1 = A \cup B$ and $t_1 = B \cup C$, then $v_1(s_1) + v_1(t_1) = 3 + 3 = 6$, which is less than $v_1(s_1 \cup t_1) + v_1(s_1 \cap t_1) = 5 + 2 = 7$, and therefore V is not submodular.

Example III.6: Let $m = 4$, and (as in example III.5) let $v_1(x)$ be a function only of the cardinality $\#(x)$ of x.

$$\begin{array}{rccccc} k = & 0 & 1 & 2 & 3 & 4 \\ v_1(s_1 : \#(s_1) = k) = & 0 & 1 & 1 & 1 & 4 \end{array}$$

and let $v_i(x) = 0$ for all x when $i > 1$.

It may be verified that $V(X)(=v_1)$ in example III.6 is normal, monotonic, and discounted. However, since $v_1(s_1 : \#(s_1)=4)$ is greater than twice $v_1(s_1 : \#(s_1) = 2)$, this value function can not be subadditive. Thus the following lemma has been verified.

Lemma III.5: For normal and monotonic functions, the following implications exist between various forms of the third condition.

$$3 \Rightarrow 3' \Rightarrow 3'' \Rightarrow 3^*, \text{ and } \underline{3} \Rightarrow \underline{3}' \Rightarrow \underline{3}'' \Rightarrow \underline{3}^*.$$

The main results of this section establish "tight" bounds on the ratio of the value of the greedy solution to that of the optimal solution. Most theorems in this section will be in two parts. The first will establish a lower bound on the ratio by considering the value of the objective function after adding the first item. An appropriate example will show how to generate ratios arbitrarily close to this bound, thus establishing the tightness of the bound.

Theorem III.1: When problem III.2 has a normal, monotonic, and discounted objective function, the greedy heuristic will result in at least $1/m$ of the value of the revenue maximizing assignment.

Proof: Since the monotonicity of the objective function implies that the greedy heuristic will never decrease a variable, the first item to be assigned will still be in the final greedy solution and the greedy solution value will be at least the value of the first item. Thus it suffices to show that the first item, which must be the most valuable single item, has a value of at least $1/m$ of the optimal solution value. This may be verified by noting that for the optimal solution X^*, and corresponding value $V(X^*)$, the restriction that the objective is discounted implies that

$$\begin{aligned} V(X^*) &\leq \textstyle\sum_{i,j:\ X^{*i}_j=1} V(e^i_j) \\ &\leq \text{maximum}_{i,j}\ V(e^i_j)\ \#(\{(i,j):\ X^{*i}_j=1\}) \\ &\leq m\ \ \text{maximum}_{i,j}\ V(e^i_j) \quad \text{since for feasible solutions,} \end{aligned}$$

$\#(\{(i,j):\ X^{*i}_j=1\}) \leq m$. Thus,

$\text{maximum}_{i,j}\ V(e^i_j) \geq V(X^*)/m$ as desired.

By the monotonicity of the objective, the greedy solution value must be at least that of the most valuable single item. The above proof verified that this item must be worth at least $1/m$ of the value of the revenue maximizing assignment. The following example shows that it is possible to construct cases in which the most valuable item is arbitrarily little more than this lower bound, and that the remaining items add arbitrarily little to the greedy solution value.

Example III.7: Let $n \geq 2$, $m \geq 2$, $0 < d < e/m$, and

$v_1(1) = 1+d$, $v_1(s_1) = d-1+\#(s_1)$ if $\#(s_1) > 1$ and $1 \in s_1$,

and $v_1(s_1) = \#(s_1)$ if $1 \notin s_1$;

$v_2(s_2) = d\ \#(s_2)$ if $1 \notin s_2$, and $v_2(s_2) = 1-d+d\ \#(s_2)$

if $1 \in s_2$; and

$v_i(s_i) = 0$ for all s_i when $i > 2$.

Note that these v_i are not only normal, monotonic, and discounted, but (like the v_i of example III.2 and III.3) the v_i are also sub-additive. The revenue maximizing assignment in example III.7 is $s_1 = (2,3,\ldots,m)$ and $s_2 = (1)$, with a resulting revenue R^* of m. The greedy heuristic will first assign item one to the first player because $\text{maximum}_{i,j}\ V(e_j^i) = V_1(e_1^1) = 1+d$. For the first player, the marginal value of any second item is now zero, whereas the marginal value to the second player is d. Thus, all the remaining items will be assigned to the second individual. This results in $s_1 = (1)$ and $s_2 = (2,3,\ldots,m)$. The resulting greedy solution revenue is $1 + md$, which by the assumption on d is less than $1 + e$.

Since $R^* = m$, the greedy revenue of $1 + e = R^*/m + e$. Thus, for any arbitrarily small positive value for e, there is an example for which the greedy solution value is less than e in excess of $1/m$ of that for the optimal solution. This proves the following theorem.

Theorem III.2: When problem III.2 has a normal, monotonic, and discounted objective funciton, then for any $e > 0$, there is an example (based on example III.7) such that the resulting greedy solution revenue R_G satisfies: $R^*/m \leq R_G < R^*/m + e$.

Results similar to the above may be obtained for the case of variably discounted functions. However, in this case, the bound must be in terms of the discounting functions. Since condition 3* (with a single discount function D) is the special case of condition <u>3</u>* where all the D_i are equal, the results will be stated in terms of the more general context.

Although the discount functions D_i may be any functions such that condition <u>3</u>* is satisfied, it will be assumed that the D_i actually considered is the minimum possible discount function. In particular, for any fixed k, there will be a finite number of subsets containing exactly k of the m items in the estate. Then let $D_i(k) = \text{maximum}_{s_i:\ \#(s_i)=k}\ v_i(s_i)/\sum_{j:\ j\epsilon s_i} v_i(j)$. Thus, a minimum function D_i exists for each player. In actual situations, this minimum discount function may be extremely difficult to calculate and some approximately minimum function may be used. The following bounds use the minimum discount function; equally correct (but not "tight") bounds result from using non-minimal discount functions.

<u>Theorem III.3</u>: When problem III.2 has a normal, monotonic, and variably discounted objective function (with discount functions D_i), then the greedy heuristic will result in a value of at least

$$1/\text{maximum}_{k_i:\ k_i \geq 0,\ \sum_{i=1}^{i=n} k_i = m,\ k_i \text{ integer}} \sum_{i=1}^{i=n} k_i\ D_i(k_i).$$

of the value of the revenue maximizing assignment.

Proof: As with the proof of theorem III.1, it is necessary only to verify that the most valuable single item has at least the desired fraction of the optimal value. This may be verified by noting that for the

optimal solution X^*, and corresponding value $V(X^*)$, the restriction that the objective is variably discounted implies that

$$V(X^*)$$

$$\leq \sum_{i=1}^{i=n} \#(\{j: X^{*i}_j=1\})\ D_i(\#(\{j: X^{*i}_j=1\}))\ \sum_{j:\ X^{*i}_j=1} V(e^i_j)$$

$$\leq \max_{i,j} V(e^i_j) \sum_{i=1}^{i=n} \#(\{j: X^{*i}_j=1\})\ D_i(\#(\{j: X^{*i}_j=1\}))$$

$$\leq \max_{i,j} V(e^i_j) \max_{k_i:\ k_i \geq 0,\ \sum_{i=1}^{i=n} k_i=m} \sum_{i=1}^{i=n} k_i\ D_i(k_i)$$

Solving the inequality for "$\max_{i,j} V(e^i_j)$" completes the proof.

The following example shows that for discount functions not uniformly bounded by one, the greedy heuristic may do arbitrarily poorly.

<u>Example III.8</u>: Consider example III.7 except that $v_1(M)$ is now equal to the variable $v^* > m$.

As v^* increases from its former value of $m - 1 + d$ to m, the revenue maximizing assignment remains unchanged. However, as soon as v^* exceeds m, there is a $d > 0$, such that $v^* > m + d$. Thus, for $v^* > m$ the value function v_1 is no longer discounted, and the optimal solution also changes to $s_1 = M$. The greedy solution remains unchanged, but is now only $(1 + md)/v^*$ of the optimal. Thus, as v^* goes to infinity, the ratio goes to zero.

Now consider the discount functions. For $i \geq 2$, D_i is bounded by one. $D_1(k) = 1$ for $k < m$, and $D_1(m) = v^*/(m+d)$. Thus, for any $v^* > m$, there is a $d > 0$ such that $D_1(m) > 1$, and the sum $\sum_{i=1}^{i=n} k_i\ D_i(k_i)$ attains its maximum when $k_1 = m$ and all other $k_i = 0$. The corresponding value for the sum is $v^*\ m/(m+d)$.

By the above theorem, the greedy heuristic results in a value of at least $(m+d)/v^* m$ of the optimal v^*. In other words, the theorem assures a greedy solution value of at least $v^* (m+d)/v^* m$ which is equal to $1 + d/m$. Notice that the actual greedy solution value in the above example is $1 + md$; for any $e > 0$, there is a $d > 0$ such that $1 + md < 1 + d/m + e$. This example, together with theorem III.3 for discounted functions proves the following theorem.

Theorem III.4: When problem III.2 has a normal, monotonic, and variably discounted (with minimum discount functions D_i) objective function, then for any $e > 0$, there is an example (based on example III.8) such that the resulting greedy solution revenue is less than the maximum of $R^*/m + e$ and $R^*/\text{maximum}_{k_i:\ k_i \geq 0,\ \sum_{i=1}^{i=n} k_i = m,\ k_i \text{ integer}} \sum_{i=1}^{i=m} k_i\, D_i(k_i)$.

Thus the lower bound on the performance of the greedy heuristic is "tight."

Although the examples III.7 and III.8 are constructed to show how poorly the greedy algorithm may do, they are motivated by an actual real world auction problem. In the motivating case [11], a bank is selling four plots of land, three contiguous and roughly similar plots, and one larger separate plot (which borders on one of the city's school properties). The bank decided to accept bids on individual plots, on the three contiguous plots as a set, and on all four plots as a set. With only little modification, this fits the scheme in which bids on all possible subsets are allowed

Consider the possibility of two potential bidders. The first is a

developer wishing to build one apartment house. The sizes of the bids on single plots reflect the sizes of the bids, and therefore the size of the apartment house which may be built there. Contiguous plots can be used to build one bigger building; thus the additive value for all subsets of the three smaller properties (in our story, the three smaller properties are pairwise adjacent). However, two non-contiguous plots are worth very little more than the most valuable of the individual plots (since the one apartment house must be built on contiguous land).

The second potential bidder, perhaps the city government, desires only the large plot; but is of course willing to pay a sufficiently small additional sum for any of the other plots. Thus, these two bidders might submit bids as in example III.7. Unfortunately, if the bank were to use the greedy heuristic in selling the land, it would sell the large plot to the developer and the three smaller plots to the second bidder. The resulting revenue would be about one fourth of that obtained from selling the large plot to the second bidder and selling the three small plots to the first bidder. This example seems plausible enough so that the negative results can not be dismissed as mere mathematical pathologies; it must be concluded that the greedy heuristic is not a satisfactory heuristic for solving the assignment problem associated with the generalized auction scheme.

III.5 A Special Case of the Auction

The previous sections present some results indicating that the general auction scheme may be infeasible in actual situations because the associated set partitioning problem is too difficult to solve. However, for any particular situation, there may be sufficient structure to the value

functions such that a solution procedure which uses this structure can solve the problem in a reasonable amount of time. This section presents a class of value functions whose structure result in set partitioning problems relatively easy to solve.

In examples III.5 and III.6, the value function of each player is a function only of the number of goods in the subset. When all items in the estate are similar then such a form of the value function appears reasonable. Stock tenders, treasury bonds, bullion sales, and wheat futures are a few examples of real world situations where the collection of goods to be auctioned consists of many identical items; the items may be divisible as in the case of some bullion sales, or indivisible as in the case of stock tenders. Thus, there are actual situations where the value functions might be assumed to be a function solely of the size of the subset.

The calculation of exact solutions using dynamic programming is considerably simpler for value functions depending only on the size of the subset than for arbitrary value functions. The following lemma shows that for the restricted form of the value functions, an exact solution may be calculated in approximately $n\ m^2/2$ elementary arithmetic calculations.

Lemma III.6: If each player's value function is a function solely of the number of items in the subset, then the revenue maximizing assignment may be calculated in order $n\ m^2$ (or approximately $n\ m^2/2$ elementary arithmetic calculations.

Proof: The procedure is identical to that used for arbitrary value functions (in Lemma III.3) except that now each iteration must only

calculate how many (rather than which) items any player should receive. In particular, each step within an iteration involves considering all possible divisions of a given number of items among the player i and the players in N_{i-1}. For each k, one must compute $v_{N_i}(k) =$

$$\max_{k_i \in \{0,1,\ldots,k\}} (v_i(k_i) + v_{N_{i-1}}(k-k_i)),$$

where $v_{N_{i-1}}(k-k_i)$ is the maximum value for assigning $k-k_1$ items among the players in N_{i-1} (this value was calculated in the previous iteration). However, this maximum may be calculated by considering all $k+1$ possible values for k_i. Thus each iteration requires $\sum_{k=0}^{k=m} (k+1) = (m+1)(m+2)/2$ calculations, which is approximately $m^2/2$ and on the order of m^2. Thus, since there is one iteration for each player, the total number of calculations is approximately $n\, m^2/2$. (In addition, at each iteration, the optimal assignment of k items, $k = 0,1,\ldots,m$ must be recorded; this requires on the order of m storage locations.)

The lemma shows that dynamic programming can take advantage of the structure when value functions depend only on the number of items in a subset.

Although the lemma is in terms of indivisible commodities, there is a similar procedure for divisible items. In this case, one must compute $v_{N_i}(k)$ for each real number in the interval $[0,1]$. Depending on the nature of the value functions v_i, the $v_{N_i}(k)$ may or may not be functions which are relatively simple to specify. If the v_{N_i} are relatively simple, then a procedure similar to that in lemma III.6 may be used.

If the value functions are restricted yet a little further, then the

greedy heuristic will produce a revenue maximizing assignment. A player is said to have a marginally decreasing value function if the value function depends only on the number of items and in addition $v_i(k+1) - v_i(k) \leq v_i(k) - v_i(k-1)$ for $k = 1,2,\ldots,m-1$. This property is very similar to subadditivity and is likely to be reasonable in many situations.

Lemma III.7: If each player's value function is marginally decreasing, then the greedy heuristic results in a revenue maximizing assignment.

Proof: Let $\underline{s}$ be a greedy solution and $\underline{s}^*$ be a revenue maximizing assignment. Let v^* be the marginal value of the item assigned last by the greedy algorithm, let k_i^* be the largest integer such that $v_i(k_i) - v_i(k_i-1)$ is greater than v^*, and let $k_{i^{**}}$ be the smallest integer such that $v_i(k_i+1) - v_i(k_i)$ is less than v^*. Then, the greedy heuristic will assign player i at least k_{i^*}, but no more than $k_{i^{**}}$ items. Furthermore, $v_i(k_i+1) - v_i(k_i) = v^*$ for all integers k_i such that $k_{i^*} < k_i \leq k_{i^{**}}$. (Since the value functions are marginally decreasing, $k_{i^*} \leq k_{i^{**}}$.)

Thus, a greedy solution may be derived by first assigning each player i k_{i^*} items and then assign the remaining $m - \sum_{j=1}^{j=n} k_i^*$ items arbitrarily so long as no player i is assigned a total of more than $k_{i^{**}}$ items. It is easy to verify that such assignments are the only ones which may arise from the greedy heuristic, and also that they are all of equal value.

Now consider the revenue maximizing assignment $\underline{s}^*$, and assume that $R(\underline{s}^*)$ is greater than the value of the greedy solutions; $\underline{s}^*$ is assumed not to be a greedy solution. Thus, there must be some player j_1 for

whom either $\#(s_j*) < k_j*$ or $\#(s_j) > k_j*$.

If $\#(s_{j_1}*) < k_{j_1}*$ then there must be a player j_2 with $\#(s_j*) > k_j*$ (this follows from the definition of the k_j*). But now, the assignment $\underline{s}'$ which assigns player j_1 one more item than in $\underline{s}^*$ and player j_2 one less will result in a revenue exceeding $R(\underline{s}^*)$. This contradicts the fact that $\underline{s}^*$ is revenue maximizing.

A similar contradiction may be obtained for the case when $(s_{j_1}*) > k_{j_1}*$. Thus $\underline{s}^*$ cannot be a revenue maximizing assignment; the revenue maximizing assignments are precisely the greedy solutions.

This lemma shows that the greedy heuristic is a particularly efficient algorithm for solving auctions with marginally decreasing value functions. Although the lemma considers only sets of indivisible goods, a similar result and algorithm clearly exists for divisible items when each value function v_i depends only on the amount of the divisible good and in addition the second derivative (with respect to the amount of goods) is non-positive.

Lemma III.6 is motivated by the possibility that the collection of goods to be auctioned consists of many identical items. A similar, but slightly less restrictive, case is when there are a small number of different types of items, and each item in the collection is of one of these types. It is then reasonable to consider value functions which are a function only of the number of items of each type that are contained in the subset. Thus, if there are k types of items, and m_j items of type j, then the value functions may be assumed to be functions from $M_1 \times M_2 \times \ldots \times M_k$ (where $M_j = \{0,1,\ldots,m_j\}$) to the real numbers.

Similar to the dynamic programming approach in lemma III.6, there is a relatively simple solution procedure for such value functions. At each iteration, one must calculate $v_{N_i}(x_1,x_2,\ldots,x_k)$ for each point $\underline{x}$ in $M_1 \times M_2 \times \ldots \times M_k$. If this is done by enumeration (as in lemma III.6), then each iteration requires $\sum_{x_1=1}^{x_1=m_1} \ldots \sum_{x_k=1}^{x_k=m_k} (\Pi_{j=1}^{j=k} x_j)$(or on the order of $(\Pi_{j=1}^{j=k} m_j)^2$) calculations. Again there are a total of n iterations, thus proving the following lemma.

<u>Lemma III.8</u>: If each players value function is a function from $M_1 \times M_2 \times \ldots \times M_k$ to the real numbers, then a revenue maximizing assignment can be calculated in approximately $n\ (\Pi_{j=1}^{j=k} \#(m_j))^2/2^k$ elementary arithmetic operations.

Thus, if there are ten items of each of three types, the number of calculations is on the order of one million operations for each player (and at each iteration, $11^3 = 1331$ intermediate solutions must be stored). A problem of this size would require only a few seconds of computer time (and a rather minimal amount of storage). Problems with three types and thirty items of each type to be allocated among ten players may be solved in approximately ten hours of computer time (and the $31^3 = 29{,}791$ intermediate solutions which must be stored at each iteration would require a small fraction of a modern computer's storage). Thus, for the cost of about ten hours of computer time (typically a few thousand dollars), problems with a hundred items may be solved if the items are of only three different types.

There are actual problems in which such a solution technique might be applicable. In a recent sale of off shore oil leases, approximately 100

sites were sold. The total revenue exceeded a billion dollars. A traditional sealed bid auction was used. It is suspected [5] that small firms hedged very strongly on their bids in order to avoid being awarded large numbers of sites. (There is one small company which apparently did not hedge enough; it was awarded "too" many sites and resold a number of them.)

Using the more general auction scheme might result in the small companies submitting more competitive bids on small sets of leases and thus in a greater total revenue. If the approximately 100 sites could be grouped into three classes of sites (those suspected of containing large amounts of oil, "medium" amounts, and "small" amounts) then dynamic programming could be used to calculate a revenue maximizing assignment. When dealing with total revenues in excess of one billion, it appears reasonable that any computer costs might be recovered from the increased revenue resulting from a more efficient assignment of the sites. (In addition, the government might realize the side benefit of having more sites awarded to smaller companies; this may result in more competition for the larger companies.) This suggests that a more general sealed bid auction scheme might be approprate in future sales of off shore oil leases.

There does not exist a result corresponding to lemma III.7 for the case where there is more than one type of item. This is illustrated by example III.2, in which there are three types of items. Since there is only one item of each type, the value functions are (trivially) "marginally decreasing" in each type of item. In addition, the value functions are subadditive; subadditivity is perhaps the most natural extension of the idea of decreasing marginal values of items of different types. Yet, the greedy heuristic does not give a revenue maximizing assignment of the goods in example III.2. Thus, the natural extension of lemma III.7 is false.

III.6 Summary

This chapter examined several results related to the general auction scheme. Unfortunately, most of the results indicate that actually using the full generality of the scheme results in very difficult mathematical problems. In particular, neither sequential auctions nor the greedy algorithm are very successful at guaranteeing revenue maximizing assignments. However, there are special cases in which the structure of the value functions enables one to relatively easily solve problems of the size encounted in actual situations. Hopefully, additional work will identify more classes of actual auctions for which satisfactory assignments may be calculated easily.

CHAPTER IV

STRATEGIC ASPECTS

IV.1 Introduction

In the previous chapters, it is assumed that not only can players accurately determine their value functions, but also that the players will accurately report these values. If there is some advantage to be gained in reporting false values, then it is quite possible that such falsification will occur.

The fairness concepts and auction schemes discussed are based on the values which the players report; so far, it has been implicitly assumed that these values accurately reflect the players' true values. Fortunately, in many situations with less than perfect information about other players' values, there is no advantage (and often a disadvantage) in reporting false values.

When it is possible for a coalition of players to communicate their values, then the interests of the coalition as a whole might result in an incentive to report false values.

Example IV.1: Three players with equal shares are to divide an estate consisting of a single painting. The players value this painting at \$30, \$27, and \$6 respectively.

Assume that the estate in example IV.1 is to be allocated according to Knaster's scheme; according to an individually-reasonable allocation with proportional division of any excess. Without collusion among the players, the painting will be awarded to the first player at a cost of \$30. The

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resulting allocation is $\underline{a}$ = ((painting - \$17), \$12, \$5).

However, if the second and third players cooperate, they could in total ignorance of the first player's bid, report false values for the painting such that the painting will still be assigned to the same player as before, and such that the second and third player will together receive "more" goods than in the previous case. In particular, the third player could bid just under \$27; for concreteness, say \$26.97.

Regardless of what the first player's bid is, the third player will still not be awarded the painting. The second player will be awarded the painting if and only if the second player receives the painting in the previous case. Thus, altering the third player's bid to \$26.97 can not affect the assignment of goods.

Using the false bid of \$26.97, there would be a smaller excess, reducing the value of the first player's award, and thus allocating more to the two cooperating players. In particular, the final allocation would be a' = ((painting - \$19.33), \$9.67, \$9.66). The first player has been "cheated" out of \$2.33.

The second player receives \$2.33 less than before, and thus may be reluctant to cooperate with the third player. However, the third player can promise to make up the difference, and can in addition afford to pay a bribe of up to \$2.33 to gain the cooperation of the first player. Even in situations where cooperation and cash side payments are acceptable it seems "unfair" that they may affect the outcome of a fair division scheme. Ideally, fair allocation schemes would not be affected by coalition formations.

The main effort in this chapter is to determine for which models of cooperation bidding honestly is a minimax strategy; and if honesty is not

minimax, how much must the honest bids be modified before the resulting strategy is minimax. Throughout this chapter the term "honest bids" may be interpreted either as each player bidding honestly, or as the coalitions joint bid v_{N_i}' (to be defined formally below) being equal to the coalitions true joint value function v_{N_i} (also defined formally below). The results usually only consider the coalitions joint bid function v_{N_i}'; however, it follows trivially from the formal definitions below that if each player bids honestly, the joint bid function will also be honest. Thus the reader is free to interpret "honest bids" either with respect to single players or with respect to coalitions of players.

Throughout this chapter, a "coalition" is a set of players which have agreed to act together. Coalitions are not dynamic; once a coalition exists, the membership of the coalition does not change. No player may belong to more than one coalition; different coalitions are disjoint subsets of the set of players N. Although coalitions are static, it is possible to talk of "sets of players" which do not necessarily form a coalition. One may usually view any set of players as a potential coalition; formally, however, a set of players is viewed only as collection of numbers (i.e., an index set) and there is no implication about the possible formation of any coalition containing these members.

If the coalitions are sufficiently ignorant of the true value functions for other sets of players, then bidding honestly is indeed a minimax strategy for the reasonable allocation scheme. As in the above example, bidding honestly need not be a minimax strategy for other individually- or overall-reasonable allocation schemes; players may have an incentive to increase their bid on the set consisting of the entire estate to the highest true value of the estate to any player in their coalition.

If coalitions have enough information about the values of other sets of players, then honest bids may no longer be minimax; if I know some other player will bid \$100 more for the entire estate than my true value, then I may safely increase my bid by almost \$100 and thereby (as in the above example) receiving a more valuable final allocation. When the estate contains more than one item, a coalition must have considerable information about other players values before they may safely overbid their true values. Some indication of how much information is needed is given in the discussion.

This chapter concludes with an analysis of equilibrium points for overall- and individually-reasonable allocations. Strict equilibrium are shown to be impossible, so attention is restricted to "almost" or δ-equilibrium points. A "weak" δ-equilibrium point is constructed for arbitrary data. Players cannot gain more than δ by unilaterally deviating from the equilibrium bids, but is "weak" in the sense that a player need not necessarily do worse by deviating from the equilibrium. Thus the resulting equilibrium points are not as self policing as they might be. However, the constructed equilibrium point results in final allocations of value very similar to reasonable allocations; no award differs in value by more than δ from that obtained under a reasonable allocation based on true values.

IV.2 Models of Cooperation

Several models of cooperation will be considered. In each, the players N will be partitioned into disjoint coalitions $N_1, N_2, \ldots, N_k$. Each coalition is assumed to contain at least one player. Unless otherwise stated, the number of coalitions k must be at least two.

In individually- and overall-reasonable allocations the excess need not be divided in proportion to players' shares f_i. Let w_i be the fraction of the excess which player i is to receive; the w_i are non-negative numbers summing to one. Now define the following.

n_i = the number of players in N_i;

$f_{N_i} = \sum_{j:\, j \in N_i} f_j$ = the share of players in N_i in the estate;

$w_{N_i} = \sum_{j:\, j \in N_i} w_j$ = the share of players in N_i in the excess;

$v_{N_i}(s)$ = the value of a revenue maximizing assignment of the goods in s to the players in N_i;

v_i' = the values (assumed to be additive in dollars) which player i actually bids;

$v_{N_i}'(s)$ = the value of a revenue maximizing assignment (based on the values the players actually bid) of the goods in s to the players in N_i;

$s_{N_i} = \bigcup_{j:\, j \in N_i} s_j$ = goods assigned to players in N_i;

$a_{N_i} = \bigcup_{j:\, j \in N_i} a_j$ = total allocation of players in N_i;

$R^{*\prime}$ = value of an assignment which maximizes revenue with respect to the players' bidded value functions v_i';

$e = v'_{N_i}(M) - v_{N_i}(M)$, where N_i should be clear from the contract.

Note that in the above definitions, the phrase "coalition N_i" was not used; the players in N_i need not represent an organized coalition. Throughout the results, proofs, and discussion, the symbol N_i does not imply that the coalition N_i exists or even may exist. In proofs of minimax strategies for the coalition N_i, a value function for the remaining

players $N\backslash N_i$ is specified; the players in $N\backslash N_i$ are not required to all be in the same coalition.

If $\underline{s}$ is a revenue maximizing assignment with respect to the players' true values v_i, then $v_{N_i}(s_{N_i})$ must be equal to $\sum_{j\in N_i} v_j(s_j)$ for any set N_i of players. The corresponding result also holds for revenue maximizing assignments based on the players' bids v_i'. Thus, if N_i is a coalition of players, then even if the players in N_i are allowed to trade goods among themselves, they have no incentive to do so if the goods are assigned according to a revenue maximizing assignment with respect to players' actual values v_i. However, the same is obviously not necessarily true for revenue maximizing assignments with respect to the players' bids v_i'.

For estates containing some divisible goods, the situation is slightly more complicated. It is assumed that there exist revenue maximizing assignments; the maximum value must be attained by some assignment. This is required whether the assignments are with respect to actual values or reported bids. For example, it would be sufficient for each v_i and v_i' to be a continuous function of each divisible good.

For any collection of value functions, the value functions of sets of players has been defined. For any fixed set N_i of players there are value functions v_i and v_i' which give rise to any possible value functions v_{N_i} and v_{N_i}'. For example, let $v_{j*}(s) = v_{N_i}(s)$ for all s for some player j^* in N_i, and let $v_j(s) = 0$ for all s for all other players (if any) in N_i. The similar verification works for the bid functions v_{N_i}'.

For any particular allocation scheme, consider the following model of cooperation.

Model IV.1: Let $N_1, N_2, \ldots, N_k$ be a partition of the players into $k > 1$ coalitions. Assume the following.

1. The players must bid zero on the empty set; thus $v_j'(\emptyset) = 0$ for all j;

2. Player j is completely ignorant of the values of player j' whenever j and j' are in different coalitions;

3. Players within the same coalition know each others true values;

4. Players within a coalition may reassign (among themselves) any goods the coalition is awarded; and

5. For each coalition N_i, the players in N_i will cooperate and specify bids v_j' so as to maximize the minimum (over all Pareto optimal allocations with respect to the bid function v_j') of the value of $v_{N_i}(a_{N_i})$. The players in N_i may make appropriate side payments to compensate any players who suffered by cooperating.

The remarkable result is that for overall-reasonable allocations, bidding honest values is a minimax strategy. Several slightly different models will also be considered later.

IV.3 Minimax Strategies

Most attention will be given to determining minimax strategies for coalitions of players when the estate is to be allocated according to the overall-reasonable allocation scheme. First, however, the case when there is only one coalition (thus $N_1 = N$) will be considered.

Lemma IV.1: If, in model IV.1, there is only one coalition $(k = 1)$, then any strategy is minimax. This is true regardless of the allocation

scheme used.

Proof: Since $N_1 = N$, the coalition N_1 will be assigned all the goods in the estate. Thus regardless of the values bid,

$v_{N_1}(a_{N_i}) = v_{N_i}(M+\$D) = R^* + D.$

This lemma shows that the case of $k = 1$ is trivial in model IV.1 and may be safely ignored hereafter.

In general, there is no assurance that there are any pure strategies which are minimax. Indeed, the classical example of a two person game, the game of matching pennies (in which player one wins if the players choose the same side of the penny, and player two wins if they choose different sides) has only strictly randomized minimax strategies. This is not the case in the problem being considered.

When the goods are to be allocated according to overall-reasonable allocations, and players behave according to model IV.1, then there are pure strategies which are minimax. Although the many verifications required may obscure the main ideas, the proofs are relatively straightforward. The first theorem establishes that no strategy, pure or mixed, can guarantee the coalition N_i more than $f_{N_i} v_{N_i}(M+\$D)$. The proof exhibits a value function $v_{N\backslash N_i}'$ which limits N_i to at most this amount regardless of the pure strategy employed by N_i. Since any mixed strategy is a convex combination of pure strategies, this will prove the desired result.

Theorem IV.1: In model IV.1 with overall-reasonable allocation schemes, there are no strategies (pure or mixed) with security level strictly greater than $f_{N_i} v_{N_i}(M+\$D)$.

Proof: Consider the following bid function $v_{N\backslash N_i}'$.

$v_{N\backslash N_i}'(M+\$D\backslash s) = v_{N_i}(M+\$D) - v_{N_i}(s)$ for all subsets s of the estate; and

$v_j'(M+\$D) = v_{N\backslash N_i}(M+\$D)$ for at least one player j in $N\backslash N_i$.

Since $\max_{j\in N_i} v_j'(M+\$D) \leq v_{N_i}'(M+\$D) \equiv v_{N\backslash N_i}'(M+\$D)$, and $\max_{j\in N\backslash N_i} v_j'(M+\$D) \equiv v_{N\backslash N_i}'(M+\$D)$, it follows that $\max_{j\in N} v_j'(M+\$D) = v_{N\backslash N_i}'(M+\$D) = v_{N_i}'(M+\$D)$.

Now, define $e(s) = v_{N_i}'(s)$, and let e^* be the maximum of $e(s)$ over all possible subsets s (note that since $e(\emptyset) = 0$, e^* must be at least zero).

By the definition of $v_{N\backslash N_i}'$, the total revenue is $v_{N\backslash N_i}'(M+\$D\backslash s) + v_{N_i}'(s) = v_{N_i}(M+\$D) + e(s)$ for any assignment which awards the goods in s to the coalition N_i. Thus $e(s) = e^*$ for any revenue maximizing assignment. For any allocation $\underline{a}$ resulting from such an assignment, the value of the subset awarded to the coalition N_i is equal to

$$
\begin{aligned}
& f_{N_i} \max_{j\in N} v_j'(M+\$D) - (v_{N_i}'(s_{N_i}) - v_{N_i}(s_{N_i})) \\
& \qquad + w_{N_i} \left(\textstyle\sum_{i=1}^{i=n} v_j'(a_j) - \max_{j\in N} v_j'(M+\$D)\right) \\
& = (f_{N_i} - w_{N_i})\, v_{N_i}'(M+\$D) + w_{N_i}(v_{N_i}(M+\$D) + e^*) - e^*. \\
& = f_{N_i}\, v_{N_i}(M+\$D) + f_{N_i}\, e(M+\$D) - e^* \\
& \leq f_{N_i}\, v_{N_i}(M+\$D).
\end{aligned}
$$

Thus the exhibited $v_{N\backslash N_i}'$ limits N_i to $f_{N_i} v_{N_i}(M+\$D)$ regardless of what pure strategy v_{N_i}' is used. Since any mixed strategy is a convex combination of pure strategies, this same $v_{N\backslash N_i}$ limits N_i to $f_{N_i} v_{N_i}(M+\$D)$ even if N_i uses a mixed strategy.

With this bound on the security level, any strategy with security level equal to this level will be a minimax strategy. Thus any pure strategy with this security level will be a pure minimax strategy. The following theorem will characterize the collection of pure strategies for model IV.1 with overall-reasonable allocations. In particular, bidding honestly is indeed a minimax strategy.

Theorem IV.2: In model IV.1 with overall-reasonable allocation schemes, the security level is $f_{N_i} v_{N_i}(M+\$D)$. When $N_i \neq N$, then

1. if $w_{N_i} \geq f_{N_i} = 0$, then v_{N_i}' is a pure minimax strategy if and only if $v_{N_i}'(s) \leq v_{N_i}(s)$ for all subsets s;

2a. if $0 < f_{N_i} < 1$ and $w_{N_i} \geq f_{N_i}$, then v_{N_i}' is a pure minimax strategy if and only if $v_{N_i}'(M+\$D) = v_{N_i}(M+\$D)$ and $v_{N_i}'(s) \leq v_{N_i}(s)$ for all other subsets s;

2b. if $0 < f_{N_i} < 1$ and $f_{N_i} > w_{N_i}$, then v_{N_i}' is a pure minimax strategy if and only if $v_{N_i}'(M+\$D) = v_{N_i}(M+\$D) = v_{N_i}(M+\$D)$, $v_{N_i}'(s) \leq v_{N_i}(s)$ for all other subsets s, and for at least one $j \in N_i, v_j'(M+\$D) = v_{N_i}'(M+\$D)$.

3a. if $f_{N_i} = w_{N_i} = 1$, then v_{N_i}' is a pure minimax strategy if and only if $v_{N_i}'(M+\$D) \geq v_{N_i}(M+\$D)$ and $v_{N_i}'(s) \leq v_{N_i}(s) + v_{N_i}'(M+\$D) - v_{N_i}(M+\$D)$ for all other subsets s.

3b. if $f_{N_i} = 1$ and $f_{N_i} > w_{N_i}$, then v_{N_i}' is a pure minimax strategy if and only if $v_{N_i}'(M+\$D) \geq v_{N_i}(M+\$D)$, $v_{N_i}'(s) \leq v_{N_i}(s) + v_{N_i}'(M+\$D) - v_{N_i}(M+\$D)$ for all other subsets s, and for at least one $j \in N_i, v_j'(M+\$D) = v_{N_i}'(M+\$D)$.

Proof: The proof is in two parts; first it is shown that all the above claimed minimax strategies have security level $f_{N_i} v_{N_i}(M+\$D)$ and

are thus, by theorem IV.1, minimax strategies. The proof is completed by exhibiting appropriate $v_{N\setminus N_i}'$ which limit N_i to less than $f_{N_i}\ v_{N_i}(M+\$D)$ whenever N_i uses a pure strategy other than those claimed to be minimax. In the proof, $R^{*\prime}$ denotes the value of a revenue maximizing assignment (with respect to the bids v_i').

I. Consider the following cases, where v_{N_i}' is any strategy claimed (by the theorem) to be minimax, and $\underline{s}$ and $\underline{a}$ are the resulting assignment and allocation.

Case I.1: if $f_{N_i} = 0$, then

$$\begin{aligned} &v_{N_i}(a_{N_i}) \\ &\quad = v_{N_i}(s_{N_i} - v_{N_i}'(s_{N_i})) + w_{N_i}(R^{*\prime} + D - \max_{j \in N} v_j'(M+\$D)) \\ &\quad \geq v_{N_i}(s_{N_i}) - v_{N_i}'(s_{N_i}) \\ &\quad \geq 0 = f_{N_i}\ v_{N_i}(M+\$D) \quad \text{as desired;} \end{aligned}$$

Case I.2: if $f_{N_i} > 0$ and $w_{N_i} \geq f_{N_i}$, then

$$\begin{aligned} &v_{N_i}(a_{N_i}) \\ &\quad = v_{N_i}(s_{N_i} - v_{N_i}'(s_{N_i})) + f_{N_i} \max_{j \in N} v_j'(M+\$D) \\ &\qquad\quad + w_{N_i}(R^{*\prime} + D - \max_{j \in N} v_j'(M+\$D)) \\ &\quad \geq v_{N_i}(s_{N_i}) - v_{N_i}'(s_{N_i}) + f_{N_i}(R^{*\prime} + \$D) \\ &\quad \geq v_{N_i}(s_{N_i}) - v_{N_i}'(s_{N_i}) + f_{N_i}\ v_{N_i}'(M+\$D) \end{aligned}$$

$$\begin{aligned} \text{either} &\quad \geq f_{N_i}\ v_{N_i}'(M+\$D) = f_{N_i}\ v_{N_i}(M+\$D) \quad \text{if } 0 < f_{N_i} < 1, \\ \text{or} &\quad = -e(s_{N_i}) + v_{N_i}'(M+\$D) \geq v_{N_i}(M+\$D) \quad \text{if } f_{N_i} = 1; \end{aligned}$$

Case I.3: if $f_{N_i} > 0$ and $w_{N_i} < f_{N_i}$, then

$$v_{N_i}(a_{N_i})$$

$$= v_{N_i}(s_{N_i} - v_{N_i}'(s_{N_i})) + f_{N_i} \max_{j \in N} v_j'(M+\$D)$$

$$+ w_{N_i}(R^{*\prime} + \$D - \max_{j \in N} v_j'(M+\$D))$$

$$\geq v_{N_i}(s_{N_i}) - v_{N_i}'(s_{N_i}) + f_{N_i} v_{N_i}'(M+\$D)$$

$$\geq f_{N_i} v_{N_i}(M+\$D) \text{ (by same argument as in I.2).}$$

Thus, all the strategies claimed to be minimax have security level of at least, and thus by theorem IV.1 exactly, $f_{N_i} v_{N_i}(M+\$D)$.

II. For any pure strategy v_{N_i}' not claimed to be minimax, consider the appropriate one of the following cases.

Case II.1: if $v_{N_i}'(M+\$D) - v_{N_i}(M+\$D) = e > 0$ (and thus $f_{N_i} \neq 1$), then consider

$v_{N\backslash N_i}'(M\backslash s) = v_{N_i}'(M) - v_{N_i}'(s) - d$ for all $s \neq \emptyset$ and where $0 < d < e(1-f_{N_i})$,

$v_{N\backslash N_i}'(\emptyset) = 0$, and

maximum $_{j \in N\backslash N_i} v_j'(M) = v_{N\backslash N_i}'(M)$.

Note that $0 \leq v_{N_i}'(M) - \max_{j \in N} v_j'(M) \leq d$. The entire estate M will be awarded to N_i, with a resulting $v_{N_i}(a_{N_i})$

$$= v_{N_i}(M) - v_{N_i}'(M) + f_{N_i}(\max_{j \in N} v_j'(M) + \$D)$$

$$+ w_{N_i}(v_{N_i}'(M) - \max_{j \in N} v_j'(M))$$

$$\leq - e + f_{N_i} v_{N_i}'(M+\$D) = d\, w_{N_i}, \text{ (which since } w_{N_i} \leq 1) \text{ is}$$

$$< e\,(f_{N_i}-1) + f_{N_i} v_{N_i}(M+\$D) + e\,(1-f_{N_i})$$

$$= f_{N_i} v_{N_i}(M+\$D);$$

Case II.2: if $e < 0$, (and thus $f_{N_i} \neq 1$), then consider

$$v_{N\backslash N_i}'(M\backslash s) = v_{N_i}'(M) - v_{N_i}'(s) \quad \text{for all} \quad s \subsetneq M,$$

$$v_{N\backslash N_i}'(M) = v_{N_i}'(M) + d \quad \text{with} \quad 0 < d < -e\, f_{N_i}, \quad \text{and}$$

$$\text{maximum}_{j\in N\backslash N_i}\ v_j'(M) = v_{N_i}'(M).$$

Note that $\max_{j\in N} v_j'(M) = v_{N_I}'(M)$ and $d + \max_{j\in N} v_j'(M) = v_{N\backslash N_i}'(M)$. The entire estate M will be assigned to $N\backslash N_i$, with a resulting

$$\begin{aligned} v_{N_i}(a_{N_i}) &= f_{N_i}(\max_{j\in N} v_j'(M) + \$D) \\ &\qquad + w_{N_i}(v_{N\backslash N_i}'(M) - \max_{j\in N} v_j'(M)) \\ &= f_{N_i}\ v_{N_i}'(M+\$D) + d\, w_{N_i} \\ &< f_{N_i}\ v_{N_i}(M+\$D) + f_{N_i}\ (1-w_{N_i})\, e \\ &\leq f_{N_i}\ v_{N_i}(M+\$D); \end{aligned}$$

Case II.3: if $f_{N_i} = 1$ and if $v_{N_i}'(s') - v_{N_i}(s') = e' > e$ for some $s' \subsetneq M$, or if $f_{N_i} < 1$, $e \leq 0$, and $e' > 0$ for some $s' \subsetneq M$, then consider

$$v_{N\backslash N_i}'(M\backslash s) = v_{N_i}'(M) - v_{N_i}'(s) - d/2 \quad \text{for all} \quad s \neq \emptyset \quad \text{or} \quad s',$$

with $0 < d < e' - \max(0,e)$,

$$v_{N\backslash N_i}'(M\backslash s') = v_{N_i}'(M) - v_{N_i}'(s') + d/2$$

$$v_{N\backslash N_i}'(\emptyset) = 0, \text{ and}$$

$$\text{maximum}_{j\in N\backslash N_i}\ v_j'(M) = v_{N\backslash N_i}'(M).$$

Note that $v_{N\backslash N_i}'(M) \leq \max_{j\in N} v_j'(M) \leq v_{N_i}'(M)$ and that $v_{N_i}'(s') + v_{N\backslash N_i}'(M\backslash s') = d/2 + v_{N_i}'(M) \leq \max_{j\in N} v_j'(M) + d$. The set s' will be assigned to N_i and the set $M\backslash s'$ to $N\backslash N_i$, with a resulting value to N_i of $v_{N_i}(a_{N_i})$

$$= v_{N_i}(s') - v_{N_i}'(s') + f_{N_i}(\max_{j\in N} v_j'(M) + \$D)$$

$$+ w_{N_i}(v_{N_i}'(s') + v_{N\backslash N_i}'(M\backslash s') - \max_{j\in N} v_j'(M))$$

$$\leq - e' + f_{N_i}(v_{N_i}'(M) + \$D) + d\ w_{N_i}$$

$$= i\ e' + f_{N_i}(v_{N_i}\ e + f_{N_i} v_{N_i}(M+\$D) + d\ w_{N_i}$$

$$\leq - e' + e + f_{N_i}\ v_{N_i}(M+\$D) + d$$

$$< f_{N_i}\ v_{N_i}(M+\$D);$$

Case II.4: if $w_{N_i} < f_{N_i}$ but $v_{N_i}'(M) > \max_{j\in N_i} v_j'(M) = v_{j*}'(M)$ for some $j* \in N_i$ (and v_{N_i}' not already covered in the above three cases), then consider,

$$v_{N\backslash N_i}'(m\backslash s) = v_{N_i}'(M) - v_{N_i}'(s) - d \quad \text{for all} \quad s \neq \emptyset \quad \text{or} \quad M,$$

with $d > 0$,

$$v_{N\backslash N_i}'(M) = v_{j*}'(M).$$

$$v_{N\backslash N_i}'(\emptyset) = 0, \text{ and}$$

$$\text{maximum}_{j\in N\backslash N_i}\ v_j'(M) = v_{N\backslash N_i}'(M)\ (= \max_{j\in N_i}\ v_j'(M)).$$

Note that $\max_{j\in N} v_j'(M) = v_{j*}'(M)$. The entire estate M will be assigned to N_i with a resulting $v_{N_i}(a_{N_i})$

$$= v_{N_i}(M) - v_{N_i}'(M) + f_{N_i}(\max_{j\in N} v_j'(M) + \$D)$$

$$+ w_{N_i}(v_{N_i}'(M) - \max_{j\in N} v_j'(M))$$

$$= - e + f_{N_i}\ v_{j*}'(M+\$D) + w_{N_i}(v_{N_i}'(M) - v_{j*}'(M))$$

$$< - e + f_{N_i}\ v_{j*}'(M+\$D) + f_{N_i}(v_{N_i}'(M) - v_{j*}'(M))$$

$$= (f_{N_i} - 1)\ e + f_{N_i}\ v_{N_i}(M+\$D)$$

$$= f_{N_i}\ v_{N_i}(M+\$D) \text{ since either } f_{N_i} = 1 \text{ or } e = 0.$$

Thus any strategy not claimed to be minimax has security level strictly less than $f_{N_i} v_{N_i}(M+\$D)$. This together with the first part, completes the proof.

The theorem shows that for reasonable allocations, honest bids are a minimax strategy. If any excess is not divided proportion to players' shares, then some $w_{N_i} < f_{N_i}$, and a minimax strategy for this (these) N_i requires that at least one player j in N_i bid $v_j'(M) = v_{N_i}'(M)$, or equivalently $v_j'(M+\$D) = v_{N_i}'(M+\$D)$. Thus for reasonable allocations, bidding honestly is a minimax strategy; for other overall-reasonable allocations the honest bids must be modified (only) slightly in order to be minimax.

IV.4 An Alternate Model

The above theorems assumes that members of N_i are allowed to reassign goods among themselves. Recall that there was no incentive for players to do so if all players bid honestly. However, to assure a minimax strategy in the case that $f_{N_i} > w_{N_i}$, some player j in N_i must bid $v_j'(M) = v_{N_i}'(M)$ which may exceed $\max_{j \in N_i} v_j(M)$. With these bids, the entire estate may be assigned to the player j with the modified bid; player j would then want to reassign the goods among the players in N_i.

It is possible to define a model in which reassignments of goods are not allowed (but sidepayments of money are). In particular, consider the following model.

Model IV.2: Consider model IV.1 except that its assumption 4 is replaced by:

4. Players may not reassign the goods awarded them. However, side payments in dollars are permissible.

This model is quite similar to the previous version and the associated minimax strategies are also quite similar. However, since players may not reassign goods, there must be some rule for deciding what assignment to use when there are several different revenue maximizing assignments. In general, it will be assumed that "random" tie breaking rules are used, where "random" implies that any of the tied assignments will be chosen with positive probability. (For estates with divisible items, there may be an infinite collection of such assignments; one possibility is to assume that any open set has positive probability and that the value functions are sufficiently well behaved in order that similar results may be derived.)

First consider the case in which $k = 1$; there is only one set N_1, and it contains all the players.

Lemma IV.2 If, in model IV.2, there is only one coalition $(k = 1)$, precisely those strategies are minimax which satisfy at least one of the following two conditions. For each assignment $\underline{s}$

$$\sum_{j \in N} v_j'(s_j) < v_N'(M), \text{ and/or}$$

$$\sum_{j \in N} v_j(s_j) = v_N(M);$$

This is true regardless of the (Pareto optimal) allocation scheme used, and gives a security level of $v_N(M+\$D) = R^*$.

Proof: The proof is in two parts.

I. Consider any collection of v_j' satisfying the lemma, and let $\underline{s}$ be the (or any, in the case of ties) resulting assignment. Since $\underline{s}$ is revenue maximizing, the first condition of the lemma can not hold for this $\underline{s}$. Thus the second condition holds and the coalition N is assigned goods valued at $v_N(M)$. Since any dollars must be allocated to players in N, the total value to N is $v_N(M+\$D)$ as required.

II. Consider any collection v_j' not satisfying the conditions of the lemma, and let $\underline{s}$ be any resulting assignment. Since it is impossible for $\sum_{j \in N} v_j'(s_j)$ to exceed $v_N'(M)$, the assumption on v_j' implies that $\sum_{j \in N} v_j'(s_j) = v_N'(M)$ and $\sum_{j \in N} v_j(s_j) < v_N(M)$. But the latter shows that the security level of such an assignment must result in a final award of value less than $v_N(M+\$D)$. Thus such strategies are not minimax.

Together, the two parts complete the proof.

As with theorem IV.1 for model IV.1, there is a corresponding result for model IV.2.

<u>Theorem IV.3</u>: In model IV.2 with overall-reasonable allocation schemes, there are no strategies with security level strictly greater than $f_{N_i} v_{N_i}(M+\$D)$.

Proof: The proof is as for theorem IV.1 except that now no reassignments are allowed; the resulting value to N_i can not exceed the value when trading was allowed. Thus the security level here can not exceed in theorem IV.1.

Using the above result, then it is clear that bidding honest values will again be a minimax strategy for N_i whenever $f_{N_i} \leq w_{N_i}$. The

following theorem corresponds to theorem IV.2; it characterizes the collection of pure minimax strategies for model IV.2 with overall-reasonable allocation schemes, and is identical to theorem IV.2 except for one additional restriction on the value functions. As in lemma IV.2, if an assignment $\underline{s}$ is not a revenue maximizing assignment (of the goods s_{N_i} among players in N_i) then the players in N_i must bid such that the sum of their bids on the s_j is less than the maximum such sum (which, presumably, corresponds to some optimal assignment of goods among players in N_i); this assures that there is no incentive for players to reassign goods.

Theorem IV.4: In model IV.2 with overall-reasonable allocation schemes and random tie breaking rules, and if $N_i \neq N$, and $w_{N_i} \geq f_{N_i}$ or $v_j(M) = v_{N_i}(M)$ for some j in N_i, then a strategy is minimax if and only if it is minimax for model IV.1 with overall-reasonable allocation schemes and in addition satisfies the following condition: for each assignment $\underline{s}$,

$$\textstyle\sum_{j \in N_i} v_j'(s_j) < v_{N_i}'(s_{N_i}) \quad \text{and/or}$$

$$\textstyle\sum_{j \in N_i} v_j'(s_j) \leq \sum_{j \in N_i} v_j(s_j) + d,$$

where d is defined to be $\max(0,e)$ (so, $d = 0$ if $f_{N_i} < 1$).

Proof: As in the proof of theorem IV.2, this proof is in two parts. In the first, it is verified that all strategies satisfying the stated conditions are indeed minimax.

I. Consider any set of bid functions v_i' claimed to be minimax. Let $\underline{s}$ be any revenue maximizing assignment. The value to N_i of the resulting allocation is then

$$\sum_{j\in N_i} v_j(s_j) - \sum_{j\in N_i} v_j'(s_j)$$

$$+ f_{N_i}(\max_{j\in N} v_j'(M) + \$D)$$

$$- w_{N_i}(R^{*\prime} - \max_{j\in N} v_j'(M))$$

Case I.1: if $w_{N_i} \geq f_{N_i}$, then the above expression is

$$\geq - e + f_{N_i}(R^{*\prime}+\$D)$$

$$\geq - d + f_{N_i}\ v_{N_i}'(M+\$D)$$

$$= f_{N_i}\ v_{N_i}(M+\$D).$$

Case I.2: if $\max_{j\in N_i} v_j(M) = v_{N_i}(M)$ and $f_{N_i} > w_{N_i}$ (thus $\max_{j\in N_i} v_j'(M) = v_{N_i}'(M)$), then the above expression is

$$\geq - d + w_{N_i}\ R^{*\prime} + (f_{N_i} - w_{N_i}) \max_{j\in N} v_j'(M) + f_{N_i}\ \$D$$

$$\geq - d + w_{N_i}\ v_{N_i}'(M) + (f_{N_i} - w_{N_i})\ v_{N_i}'(M) + f_{N_i}\ \$D$$

$$= - d + f_{N_i}(v_{N_i}'(M) + \$D)$$

$$= f_{N_i}(v_{N_i}(M) + \$D) \quad \text{since either } d = 0, \text{ or } d = e \text{ and } f_{N_i} = 1.$$

Thus any strategies statisfying the conditions of the theorem have a security level of at least (and thus by theorem IV.3, exactly) $f_{N_i}\ v_{N_i}(M+\$D)$.

II. Strategies which are not minimax in model IV.1 need not be considered again since theorem IV.2 already shows that such strategies have security level less than the desired. Consider any strategy which is minimax in model IV.1 but not claimed minimax in model IV.2. Then, for some assignment s', $\sum_{j\in N_i} v_j'(s'_j) \geq$ (and therefore exactly equal to) $v_{N_i}'(s_{N_i}')$ and $\sum_{j\in N_i} v_j'(s_j') - \sum_{j\in N_i} v_j(s_j') - d > 0$.

Now consider the following bid function for $N\backslash N_i$.

$$v_{N\backslash N_i}'(M\backslash s) = v_{N_i}'(M) - v_{N_i}'(s) \quad \text{for all } s, \text{ and}$$

$$v_j'(M) = v_{N\backslash N_i}'(M) \quad \text{for some } j \text{ in } N\backslash N_i.$$

Thus, s_{N_i}' may be assigned to N_i, and it may be assumed that s_j' is assigned to player j for all j in N_i. Now

$$\begin{aligned}
& v_{N_i}(a_{N_i}) \\
&= \textstyle\sum_{j\in N_i} v_j(s_j') - \sum_{j\in N_i} v_j'(s_j') \\
&\quad + f_{N_i} \max_{j\in N} v_j'(M+\$D) + w_{N_i}(R^{*\prime} - \max_{j\in N} v_j'(M)) \\
&< - d + f_{N_i} v_{N_i}(M+\$D) + w_{N_i}'(M) - v_{N_i}'(M)) \\
&= - d + f_{N_i} v_{N_i}'(M) + f_{N_i} \$D \\
&= f_{N_i}(v_{N_i}(M) + \$D) \quad \text{since either } d = 0, \text{ or } d = e \text{ and}
\end{aligned}$$

$f_{N_i} = 1$. Thus the strategy results in $f_{N_i}(a_{N_i}) < f_{N_i}(R^*+\$D)$.

In particular, if $w_{N_i} \geq f_{N_i}$ or $v_j(M) = v_{N_i}(M)$ for some j in N_i, then bidding honestly is a minimax strategy for the coalition N_i in model IV.2 with overall-reasonable allocations.

The above theorem does not cover all possible combinations of f_{N_i}, w_{N_i}, and true value functions. When $f_{N_i} > w_{N_i} \geq 0$ and $\max_{j\in N_i} v_j(M) < v_{N_i}(M)$ then any minimax strategy in theorem IV.2 has $v_{j^*}'(M) = v_{N_i}(M)$ for some j^* in N_i, and $v_{N_i}'(M) = d + v_{N_i}(M)$ (where $d = \max(0,e)$). Thus, $v_{j^*}'(M) = v_{N_i}(M) - v_{N_i}(M) + d > d + \max_{j\in N_i} v_j(M) \geq d + v_{j^*}(M)$. Simplifying, $v_{j^*}'(M) > v_{j^*}(M) + d$. In order to satisfy the additional conditions given in theorem IV.4, the

first choice must hold; but $v_{j*}(M) < v_{N_i}(M)$ contradicts the assumption $v_{j*}'(M) = v_{N_i}'(M)$ of theorem IV.2.

Thus, when $f_{N_i} > w_{N_i} \geq 0$ and $\max_{j \in N_i} v_j(M) < v_{N_i}(M)$, there is no pure strategy satisfying all the conditions of theorems IV.2 and IV.4. For any strategy in this situation, the combined results of the two theorems show that the security level can not be as great as $f_{N_i} v_{N_i}(M+\$D)$. However, the next theorem shows that for any positive δ, there are strategies which have a security level of at least $f_{N_i} v_{N_i}(M+\$D) - \delta$.

The theorem exhibits some strategies which are almost minimax. A strategy will be called δ-minimax if it has security level L and there is no other strategy with a security level exceeding $L + \delta$. The δ-minimax strategies exhibited below are quite similar to previous minimax strategies.

<u>Theorem IV.5</u>: In model IV.2 with overall-reasonable allocation schemes and random tie breaking rules, a strategy for $N_i \neq N$ is a δ-minimax strategy $(\delta > 0)$ if it satisfies all the following:

1. the strategy satisfies the conditions of theorem IV.2, except if $f_{N_i} > w_{N_i}$ then $v_j'(M) \geq v_{N_i}'(M) - \delta/f_{N_i}$ for at least one $j \in N_i$ (if $f_{N_i} = 0$, then replace "δ/f_{N_i}" by any non-negative constant) and

2. $\sum_{j \in N_i} v_j'(s_j) < v_{N_i}'(s_{N_i})$ and/or

   $\sum_{j \in N_i} v_j'(s_j) \leq \sum_{j \in N_i} v_j(s_j) + d$, $(d = \max(0,e))$.

Proof: Notice that the first condition is very similar to the conditions of theorem IV.2. In the cases 2b and 3b, the condition has been relaxed just enough to assure the existence of strategies. The proof

requires only verification that the strategies have the desired security level. (It is easy, though not required by the theorem, to show that strategies v_j' do always exist.)

If $f_{N_i} \leq w_{N_i}$, then the desired proof is case I.1 of the proof to theorem IV.4. When $f_{N_i} > w_{N_i}$, then (minimizing case I.2 of the proof to theorem IV.1) $\max_{j \in N_i} v_j'(M)$ must be at least $v_{N_i}'(M) - d/f_{N_i}$, and the resulting $v_{N_i}(a_{N_i})$

$$\geq - d + w_{N_i} R^{*\prime} + (f_{N_i} - w_{N_i}) \max_{j \in N} v_j'(M) + f_{N_i} \$D$$

$$\geq - d + w_{N_i} v_{N_i}'(M)$$

$$+ (f_{N_i} - w_{N_i}) (v_{N_i}'(M) - \delta/f_{N_i}) + f_{N_i} \$D$$

$$= - d + f_{N_i} v_{N_i}'(M) - (f_{N_i} - w_{N_i}) \delta/f_{N_i} + f_{N_i} \$D$$

$$\geq - d + f_{N_i} v_{N_i}'(M) - \delta + \$D$$

$$= f_{N_i} v_{N_i}(M) + f_{N_i} \$D - \delta$$

$$= f_{N_i} v_{N_i}(M + \$D) - \delta$$

(where, recall, if $f_{N_i} = 0$, then "δ/f_{N_i}" was replaced by any non-negative constant). Thus, all the strategies claimed to be δ-minimax have the desired security level.

Theorems IV.3, IV.4, and IV.5 show that minimax strategies for model IV.2 are quite similar to those in model IV.1. Basically, the only modification necessary was to ensure that players in N_i bid such that they had no desire to reassign the goods in the resulting revenue maximizing assignment. Notice that the additional restraint in theorem IV.4, exactly the same as the second condition in theorem IV.5, is always satisfied for

bids which are honest. Thus, as in model IV.1, only in the case that $f_{N_i} > w_{N_i}$ are honest bids not minimax; in that case the best strategies may require one player to overbid on the entire estate.

Although honest bids are a minimax strategy in the above models with reasonable allocation schemes, there may be other minimax strategies which do at least as well, and sometimes do better. For instance, if for the coalition N_i, f_{N_i} is zero, then any strategy which assures this coalition of receiving an allocation of non-negative value is minimax. Thus, honest bids is a minimax strategy since it assures a zero value.

Underbidding each item slightly is also a minimax strategy; if the players in N_i receive any goods, they will have made a positive profit. This minimax strategy may occasionally result in N_i receiving an allocation of positive value. Honesty is therefore not the "best" minimax strategy...indeed, it is the "worst" since using it prevents the players in N_i from ever receiving an allocation with positive value.

Since the coalitions are assumed to be totally ignorant of all other players' values, it is impossible to characterize the "best" minimax strategy. One alternative is to consider the collection of undominated minimax strategies; ruling out the strategy of bidding honestly. A different alternative will be considered below; the case of partial information will be examined.

Although the models assumed that the coalition N_i was totally ignorant of the values of other coalitions, the full strength of this assumption is never used. It is only required that the players in N_i believe that there is some chance that the remaining players will submit bids resulting in the $v_{N\backslash N_i}'$. In all of the proofs, the required $v_{N\backslash N_i}'$

are such that $v_{N\backslash N_i}'(M\backslash s) + v_{N_i}'(s)$ is an approximately constant function of s; and even this constancy is really only required for a small number of s in each proof. Thus, in a wide class of situations, it is reasonable to expect that these conditions are likely to be satisfied and so even with some information about other coalitions' bids, N_i still considers the required $v_{N\backslash N_i}'$ quite possible. Unless the players in N_i have sufficient information to rule out the possibility of the necessary $v_{N\backslash N_i}'$, the previous proofs are still valid (Note that knowing the desired $v_{N\backslash N_i}'$ is impossible does not necessarily change the results; other $v_{N\backslash N_i}$ can serve equally well in the proofs and all such bid functions must be ruled out before the results are affected).

IV.5 Partial Information

Whenever a coalition N_i believes it is possible for the highest bid among the remaining players on an estate consisting of a single indivisible item to be at least as high as the highest value for the estate by any player in N_i, then a minimax strategy for N_i is to bid honestly. If however, it is known that all bids by players in $N\backslash N_i'$ will be at least \$100 less than the highest value by players in N_i, then honest bidding is no longer a minimax strategy.

For example, if two individuals are to divide equally an estate consisting of a single painting which they value at \$200 and \$100 respectively, then if the first player knows the second player will bid $v_2' = \$100$, a δ-minimax strategy for the first player is to bid v_1' such that $100 < v_1 \leq 100 + \delta$. By bidding such a v_1', the first player receives the painting at a cost of just barely over \$50. With honest bids, the first player receives the painting at a cost of \$100, of which some may

be returned (depending on the allocation scheme). For reasonable allocations, using the available information enables the second player to profit by \$50; for individually-reasonable allocations with proportional division of any excess, the profit is \$25. In either case, bidding honestly is not a minimax strategy.

If it is only the second player who has information about the other player's bid, and the second player knows that the first player will bid $v_1' = \$200$, then for reasonable allocation schemes any bid $v_2' < \$200$ is minimax. Any such bid will result in the same final overall-reasonable allocation as with honest bids. Notice that once the players have announced their bids $v_1' = \$200$ and $\$200 - \delta < v_2' < \$200$, then neither player can unilaterally change a bid and profit by at least δ. Such δ-equilibrium points will be discussed further in a later section.

In the previous models, it is assumed that the coalition N_i is totally ignorant of the bids $v_{N\backslash N_i}'$. The following two models relax this assumption.

Model IV.3 (IV.4): Consider model IV.1 (IV.2) except that assumption 2 is replaced by the following.

2. For each set of players N_i, there is a set $V(N_i)'$. These sets are known to all players, and it is known that the bid function v_{N_i}' must be an element of $V(N_i)'$. Other than this knowledge about the sets $V(N_i)'$, the players in different coalitions are completely ignorant about what each other will actually bid.

As illustrated in the discussion, there are problems of the type considered in model IV.3 and IV.4 in which the players should take advantage of their information and not necessarily bid honestly.

Define the following two quantities.

$B(N_i)$ = the least upper bound on the set of numbers

$\{x:$ for some s and for all $v_{N\backslash N_i}'$ in $V(N\backslash N_i)'$,

$v_{N_i}(s) + v_{N\backslash N_i}'(M\backslash s) \geq x\}$; and

$G(N_i)$ = the least upper bound on the set of numbers

$\{x:$ for all nonempty s and for all $v_{N\backslash N_i}'$ in $V(N\backslash N_i)'$,

$v_{N_i}(s) + v_{N\backslash N_i}'(M\backslash s) \geq v_{N\backslash N_i}'(M)\}$.

Roughly speaking, $B(N_i)$ is (a bound on) the smallest value for the entire estate which N_i (based on the available knowledge) knows is possible when N_i bids honestly. $G(N_i)'$ is (a bound on) the largest value that N_i can subtract from each of its true values while certain that doing so will not alter the assignment of goods from that of a revenue maximizing assignment based on the (uncertain) bids $v_{N\backslash N_i}'$ and honest values v_{N_i}.

Thus, with overall- and individually-reasonable allocations, the coalition N_i knows that bidding honestly will result in a value $v_{N_i}(a_{N_i})$ of at least $f_{N_i}(B(N_i) + \$D)$. If $G(N_i)$ is positive, then N_i may be certain of receiving $v_{N_i}(a_{N_i}) \geq f_{N_i}(B(N_i) + \$D) + (1-f_{N_i})(G(N_i) - \delta)$ if N_i bids v_{N_i}' identically equal to $v_{N_i} + G(N_i) - \delta$. Although the formal, and lengthy, details will not be presented, it is plausible that if $G(N_i) \leq 0$ or if $f_{N_i} = 1$, then bidding honestly is still a minimax strategy and that the minimax strategies have a security level of $f_{N_i}(B(N_i) + \$D)$; whereas if $G(N_i) > 0$ and $f_{N_i} \neq 1$, then bidding $v_{N_i}'(s) = v_{N_i}(s) + G(N_i) - \delta$ is a δ-minimax strategy and the security level is $f_{N_i}(B(N_i) + \$D) + (1-f_{N_i})(G(N_i) - \delta)$. Thus if $G(N_i)$ is

positive and N_i has less than a complete share in the estate, the coalition N_i must bid a slightly modified version of their true values in order to be bidding minimax.

IV.6 Individually-Reasonable Allocations

The discussion at the beginning of this chapter indicates that bidding honestly is in general not a minimax strategy if individually-reasonable allocations are used. Although the main concern is to consider strategies when overall-reasonable (and particularly, reasonable) allocations are used, the results in the above sections give a very good intuitive indication for what strategies are minimax when using individually-reasonable allocations.

Honesty is in general not a minimax strategy for individually-reasonable allocations since *fairness is defined in terms of* each individual's value for the entire estate. If the individuals in a particular coalition have differing values for the entire estate, then they have an incentive to increase some of their bids above their true values. In particular, in model IV.1, it is clear that the minimax strategies for individually-reasonable allocations are those of overall-reasonable allocations which also satisfy a condition of the nature that $v_k'(M)$ is equal to the $\max_{j \in N_i} v_j'(M)$ for all players k in N_i. In some cases, for example if no reassigning is allowed, this condition may have to be modified slightly. Nonetheless, it does indicate the degree to which minimax strategies for overall-reasonable allocations must be restricted in order to be minimax for individually-reasonable allocations.

IV.7 Equilibrium Points

A previous example illustrates that there may not be any equilibrium points for fair allocation schemes. However, for overall- and individually-reasonable allocations, there do exist sets of pure strategies which are almost in equilibrium. In particular, a set $(v_j')_{j=1}^{j=n}$ of strategies will be called a set of δ-equilibrium strategies if an individual can profit by at most δ when deviating unilaterally from this stategy.

Theorem IV.6: For any revenue maximizing assignment (with respect to true values) $\underline{s}$, there is a set of δ-equilibrium strategies $(v_j')_{j=1}^{j=n}$ which, for any $\delta > 0$,

1. has $\underline{s}$ a revenue maximizing assignment (with respect to the equilibrium bids v_j'), and

2. results in an allocation with value within δ (for each player) of that for any reasonable allocation with respect to players' true values.

Proof: Let $\underline{s}$ be a revenue maximizing assignment with respect to players' true values. Consider any ordering of the players; the proof uses the order $1,2,\ldots,n$.

Let $v_i^*(M) = R^*$ for all players i, and then iteratively, let $v_i^*(s) = R^* - v_{N\backslash i}^{(i)}(M\backslash s)$ for all s such that $\emptyset \neq s \neq M$, where $v_{N\backslash i}^{(i)}(x)$ is the value of a revenue maximizing assignment of the goods x among players in $N\backslash i$ according to the values $v_1^*,\ldots, v_{i-1}^*$, $v_{i+1},\ldots,v_n$. It is assumed that this maximum value will be attained by some appropriate assignment (which is true trivially for estates consisting only of indivisible items).

Intuitively, the v_i^* are derived by starting with true values, increasing each $v_i^*(M)$ to R^*, and then iteratively increasing $v_i^*(s)$

(for each nontrivial s) as much as possible without creating an assignment with value in excess of R^*. At each stage, the values obtained in previous iterations are used in determining the next player's bid function v_i^*.

Now define v_i' by letting

$$v_i'(\emptyset) = 0, \; v_i'(s_i) = v_i^*(s_i)$$

(where $\underline{s}$ was specified above), and $v_i'(s) = v_i^*(s) - \delta/n$ for all other subsets s. It is easy to verify that these v_i' are indeed a set of δ-equilibrium strategies. If a player deviates unilaterally from these strategies, then deviations of less than δ do no affect the outcome by as much as δ; increasing a $v_j'(s)$ by more than δ results in changing the assignment to one in which player j receives the set s at a cost exceeding the true value of the set of player j; and finally, the player will not receive any set for which the bid has been decreased below $v_j'(s)$. The straightforward details of the verifications are left to the reader.

The set of δ-equilibrium strategies constructed in the proof has several nice properties. First, and in general not true for equilibrium strategies, is the Pareto optimality of the resulting allocation. In addition, the strategies are in equilibrium for a wide variety of allocation schemes; the excess never exceeds δ and all players bid approximately R^* for the entire estate M. Thus, any overall- or individually-reasonable allocation awards a player very nearly the same value. Finally, the resulting allocation is always almost a reasonable allocation.

The theorem states that there is at least one set of δ-equilibrium points for any Pareto optimal assignment of the estate. The proof uses

a particular order of the players, suggesting that different orders of the players result in different sets of strategies. Indeed, different orders may result in different strategies; the final value awarded to each player is (essentially) the same regardless of which set of strategies is used since all the strategies result in values very similar to those of a reasonable allocation based on true values.

The following examples illustrate these sets of equilibrium points; the first is used to illustrate the process by which the strategies are calculated, and the second shows that different orderings of players may indeed result in different strategies.

Example IV.1: Three players are bidding on two items. The value functions are given below.

$x =$	$\emptyset$	A	B	$A \cup B$
$v_1(x) =$	0	3	4	6
$v_2(x) =$	0	7	6	10
$v_3(x) =$	0	5	8	12

In order to compute the strategies, it is best to rewrite the above data in the form of players' values for assignments rather than sets (even though the functions will still depend only on the set any particular player is awarded). In particular, write the data as below. Let $\underline{s} = (s_1, s_2, s_3)$.

$s_1 =$	$A \cup B$	A	A	B	$\emptyset$	$\emptyset$	B	$\emptyset$	$\emptyset$
$s_2 =$	$\emptyset$	B	$\emptyset$	A	$A \cup B$	A	$\emptyset$	B	$\emptyset$
$s_3 =$	$\emptyset$	$\emptyset$	B	$\emptyset$	$\emptyset$	B	A	A	$A \cup B$
$v_1(s_1) =$	6	3	3	4	0	0	4	0	0
$v_2(s_2) =$	0	6	0	7	10	7	0	6	0
$v_3(s_3) =$	0	0	8	0	0	8	5	5	12
$v_N(A \cup B) =$	6	9	11	11	10	15	9	11	12
						$\Vert$ R^*			

Consider the order 1,2,3 of the players. Then, $v_1^*(A \cup B) = 15$, and all of the remaining $v_1(s)$ are increased until any further increase would make the sum of $v_1^*(s) + v_{(23)}(M \backslash s)$ exceed 15. The corresponding function $v_1^*(s)$ is given below.

$$v_1^*(s_1) = 15 \quad 7 \quad 7 \quad 8 \quad 0 \quad 0 \quad 8 \quad 0 \quad 0$$

and the resulting sum $v_1^*(s) + v_{(23)}(M \backslash s)$ is

$$\text{sum} = 15 \quad 13 \quad 15 \quad 15 \quad 10 \quad 15 \quad 12 \quad 11 \quad 12$$

Notice that although the second column sum is 13 (which is less than $R^* = 15$), player one can not increase $v_1^*(A)$ above 7 without making the third column sum exceed 15.

Replace the $v_1(s_1)$ row of the original data by the above row of $v_1^*(s_1)$ and repeat the above procedure for player two. Finally, replace $v_2(s_2)$ by $v_2^*(s_2)$ and repeat the procedure for the third player. The resulting bids v_j' are given below.

$v_1'(s_1) =$	15-	7-	7-	8-	0	0	8-	0	0
$v_2'(s_2) =$	0	8	0	7	15-	7	0	8	0
$v_3'(s_3) =$	0	0	8	0	0	8	7-	7-	15-
$v_N'(A \cup B) =$	15-	15-	15-	15-	15-	15	15=	15-	15-

Subtracting $\delta/3$ from each entry followed by a minus sign (subtract $2\delta/3$ from the 15 followed by "=") gives the desired set of δ-equilibrium strategies. Notice that the resulting assignment is $(\emptyset,A,B)$ (fourth column from right); the same as for the honest bids. Also, the resulting allocation has value (for each player) within δ of a reasonable allocation according to honest values.

Example IV.2: (previously, examples II.3 and III.1) Two players have equal shares in an estate consisting of two indivisible items (A and B). The value functions are as given below.

$x =$	$\emptyset$	A	B	$A \cup B$
$v_1(x) =$	0	5	7	9
$v_2(x) =$	0	2	5	9

In this example $R^* = 10$; the revenue results from the assignment $\underline{s} = (A,B)$. If the players are ordered as numbered, then the resulting v_j' are given below (where a minus sign denotes subtracting $\delta/2$).

$x =$	$\emptyset$	A	B	$A \cup B$
$v_1'(x) =$	0	5	8-	10-
$v_2'(x) =$	0	2-	5	10-

A different set of strategies is obtained by ordering the players so that player two is first. In particular, the v_j' are given below.

$x =$	$\emptyset$	A	B	$A \cup B$
$v_1'(x) =$	0	5	7-	10-
$v_2'(x) =$	0	3-	5	10-

Although the strategies are different, they result in the same final allocation (A,B) as the reasonable allocation based on the honest bids. Thus δ-equilibrium strategies need not be unique.

IV.8 Summary

This chapter examined minimax strategies for several different models of cooperation, and finally constructed a set of δ-equilibrium strategies. Although minimax strategies have been examined in considerable detail, no mention has been made of other types of strategies. Because of their relative simplicity, minimax strategies are often studied before considering more complex cases such as utility maximizing strategies.

Utility maximizing strategies are perhaps more applicable to actual problems. However, a study of these requires some further restrictions on the problem. The analysis must be in terms of utilities rather than values. In addition, the analysis must specifically use the specification of what information on the players' bids is available to each player. Although the analysis will be considerably more involved, one area in which much work remains to be done is the study of utility maximizing strategies for allocation schemes based on the general auction.

Finally, throughout the paper, it has been implicitly assumed that players know exactly their own values for all possible subsets of items.

If players are uncertain as to their bids, then they must underbid their estimated values in order to avoid the "winner's curse" (the winner tends to be the individual who most overestimated the true value of an item). Capen, Clapp, and Campbell [4] derived results for auctions of a single item. The general auction scheme is considerably more complicated, but similar results for such auctions would certainly be of great use in actual applications.

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Auctions and fair division problems are situations in which commodities are to be allocated fairly and efficiently. While a variety of schemes exist for fairly allocating finely divisible homogeneous commodities, most schemes are not applicable to the problem of allocating indivisible items. This paper considers the problem of fairly allocating sets of indivisible objects.

"Dollars," a finely divisible, homogeneous, transferrable commodity,

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are used to evaluate individuals' preferences and to transfer value among individuals. This introduction of dollars has several implications; the main result is that fair allocation problems may be viewed as two smaller problems. First auction the goods among the individuals and then divide the resulting revenue according to the chosen definition of fairness.

Several existing fair allocation schemes are reviewed; examples illustrate some difficulties associated with their use. Kuhn's definitions of "fairness" are presented and two extensions are considered for the case where individuals have different shares in the collection of goods. This chapter concludes by discussing a scheme of Dubins which permits individuals to express preferences on how any goods they do not receive are assigned among the remaining individuals.

The second chapter considers several definitions of fairness. Some define a fair share in terms of a fraction of the value for the entire estate, while others let players compare "what I get" to "what you get." One definition is closely related to many of the remaining definitions; it is singled out for further study.

Auctions are part of the fair allocation schemes considered. A general form of a sealed bid auction is examined in the third chapter. This auction does not require individuals' value functions to be additive in all goods. However, it results in a set-partitioning problem which is extremely difficult to solve.

For small numbers of players, solutions may be obtained by using dynamic programming. For large problems, either a heuristic must be used, or some less general auction scheme must be used. Some existing results on the performance of the "greedy" heuristic are extended to a slightly restricted version of the auction problem; a "tight" bound for the performance of the heuristic relative to the optimal solution is one over the number of goods to be assigned. This result provides insight into the performance of the greedy heuristic when applied to general integer programs. This chapter concludes by presenting a class of value functions for which it is relatively easy to calculate exact solutions.

The final chapter deals with some game theoretic aspects of the fair allocation schemes under several different models of cooperation among players. Bidding honestly is a minimax strategy in a variety of situations; for situations where honest bidding is not minimax, there are slightly modified forms of honest bids which are. The discussion concludes by constructing a collection of weak delta-equilibrium points which result in allocations very similar to those based on honest bids and thus appear particularly reasonable.

Finally, suggestions are made for alternate objectives which may be considered when deriving strategies for coalitions, and additional possible directions for furture research are indicated.